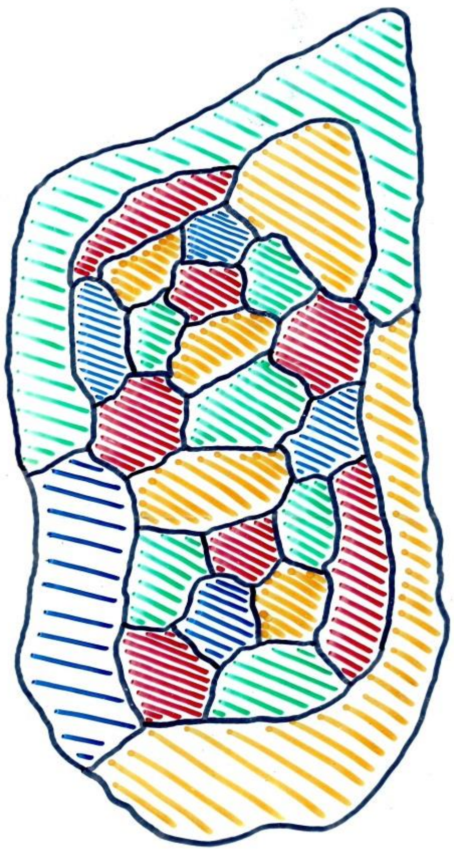
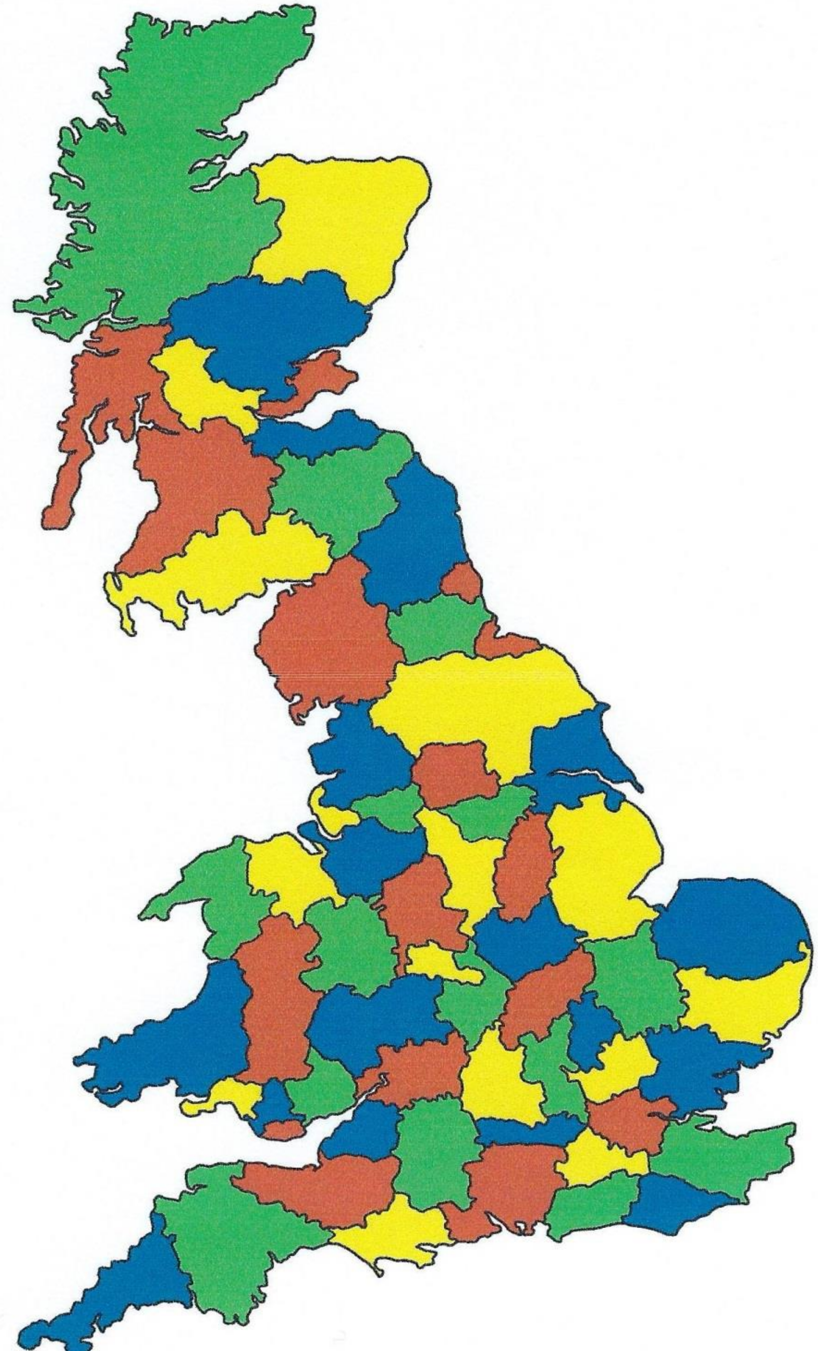
Four colours suffice

Robin Wilson





Four col~~u~~ours suffice

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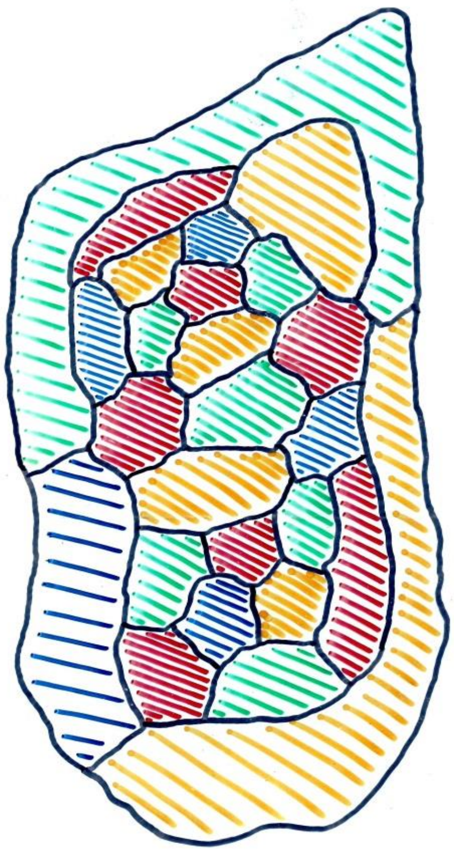
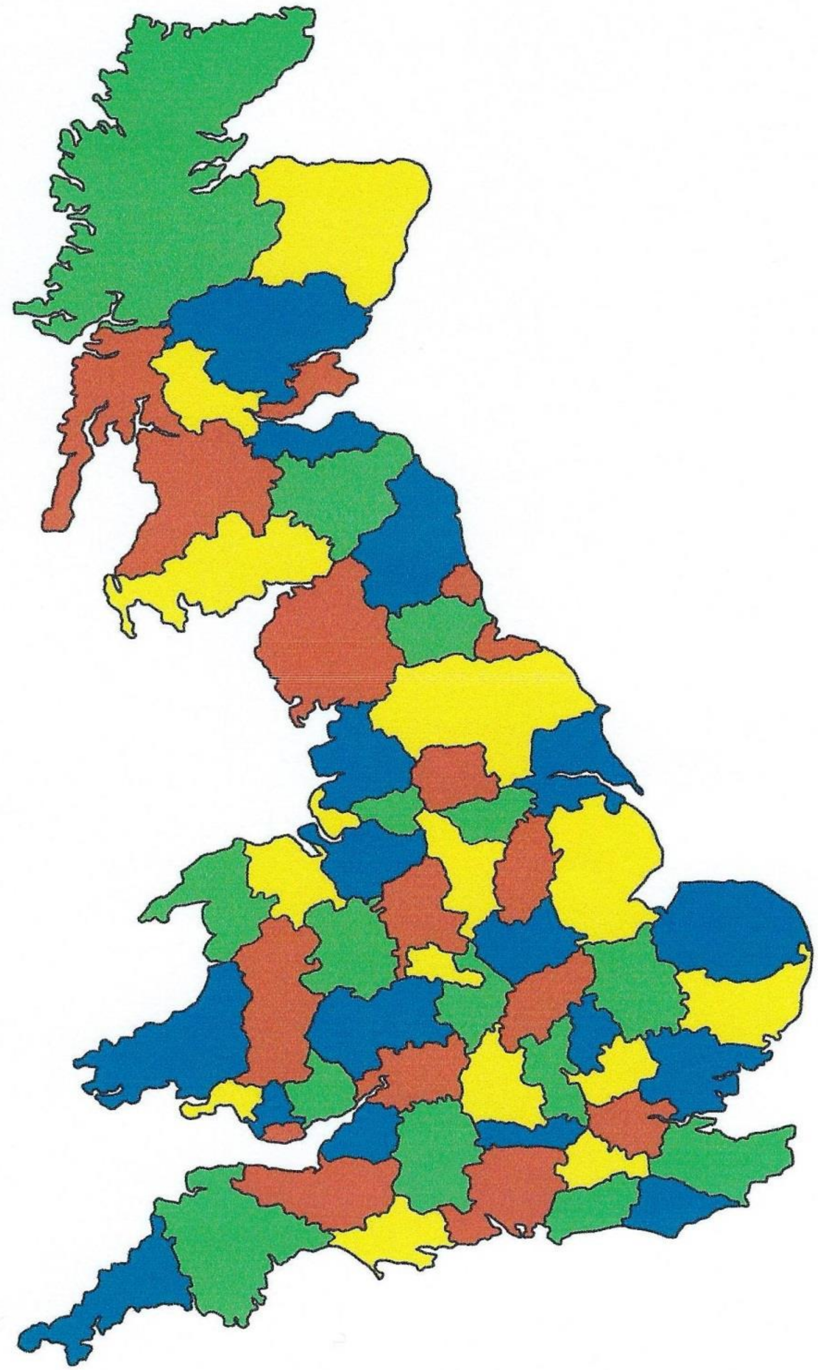
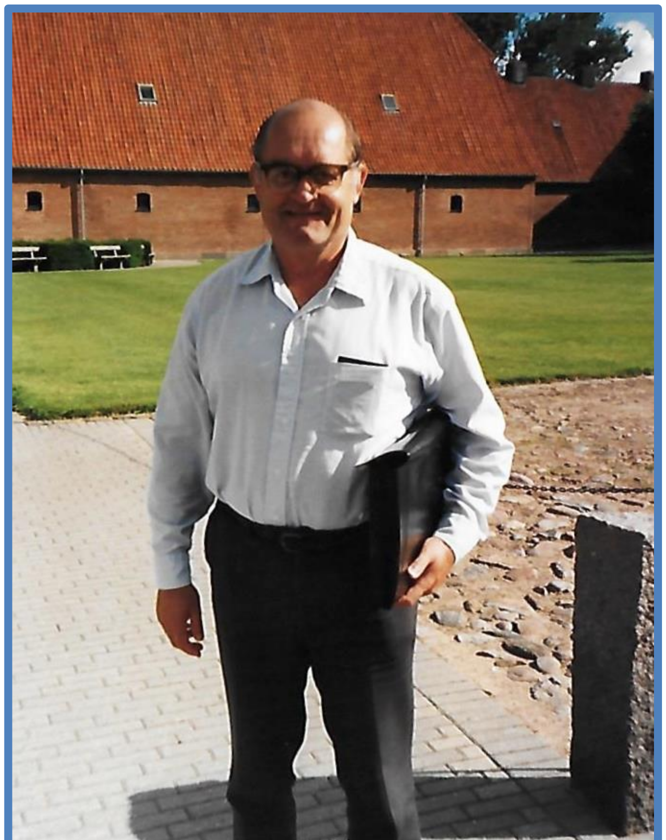
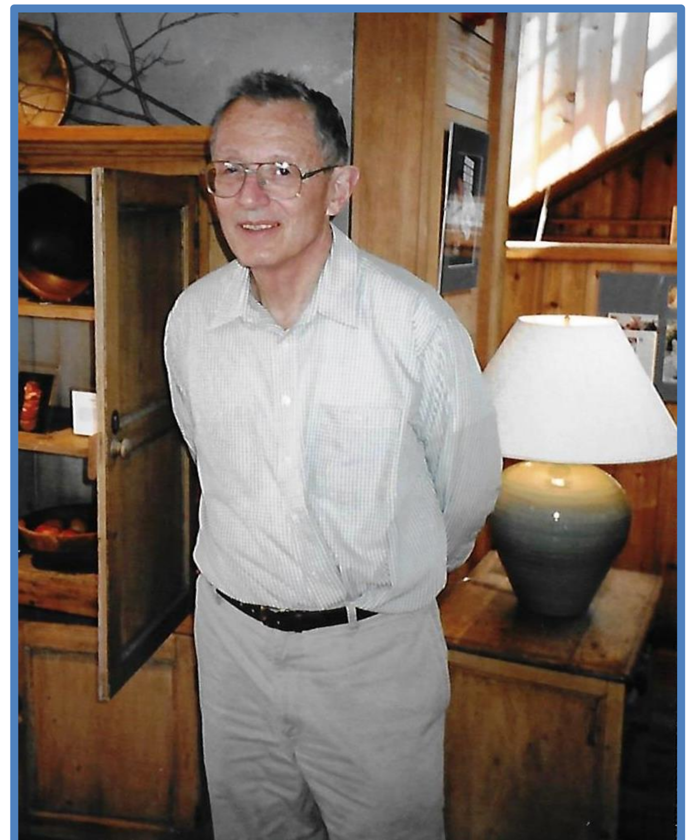




This talk is dedicated to Wolfgang Haken and the late Kenneth Appel



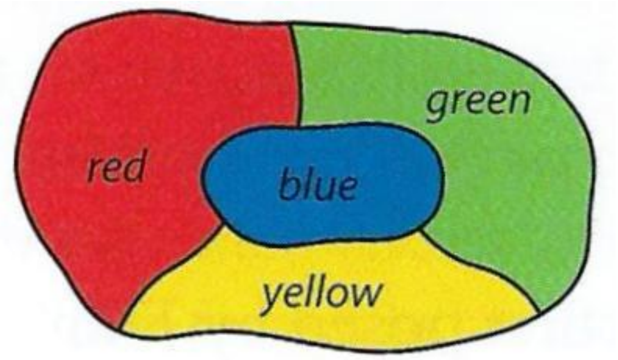


Guthrie's map-color problem

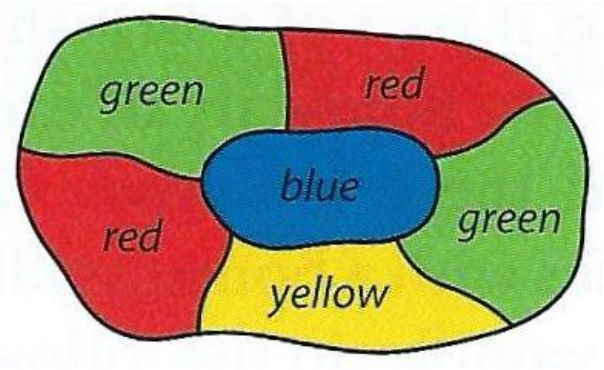
Francis Guthrie

Can every map be colored with four colors so that neighboring countries are colored differently?

We certainly need four for *some* maps



four neighboring countries



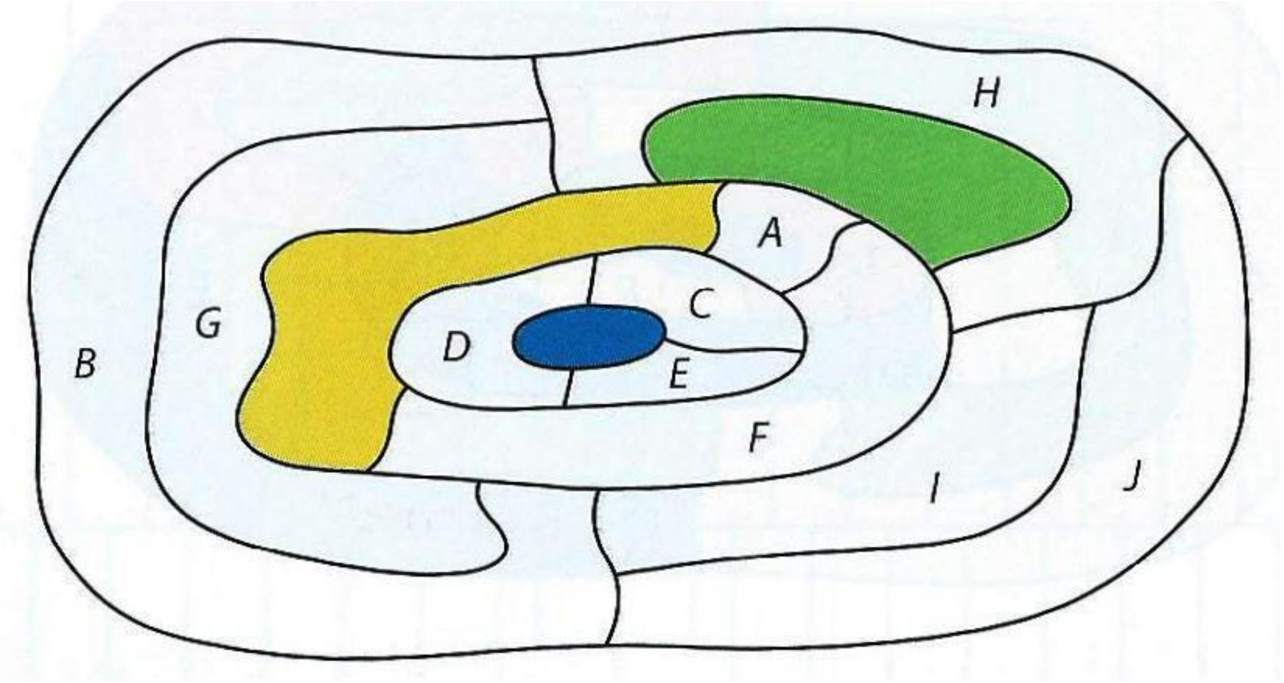
. . . but not here

. . . but do four colors suffice for *all* maps?

A map-coloring problem

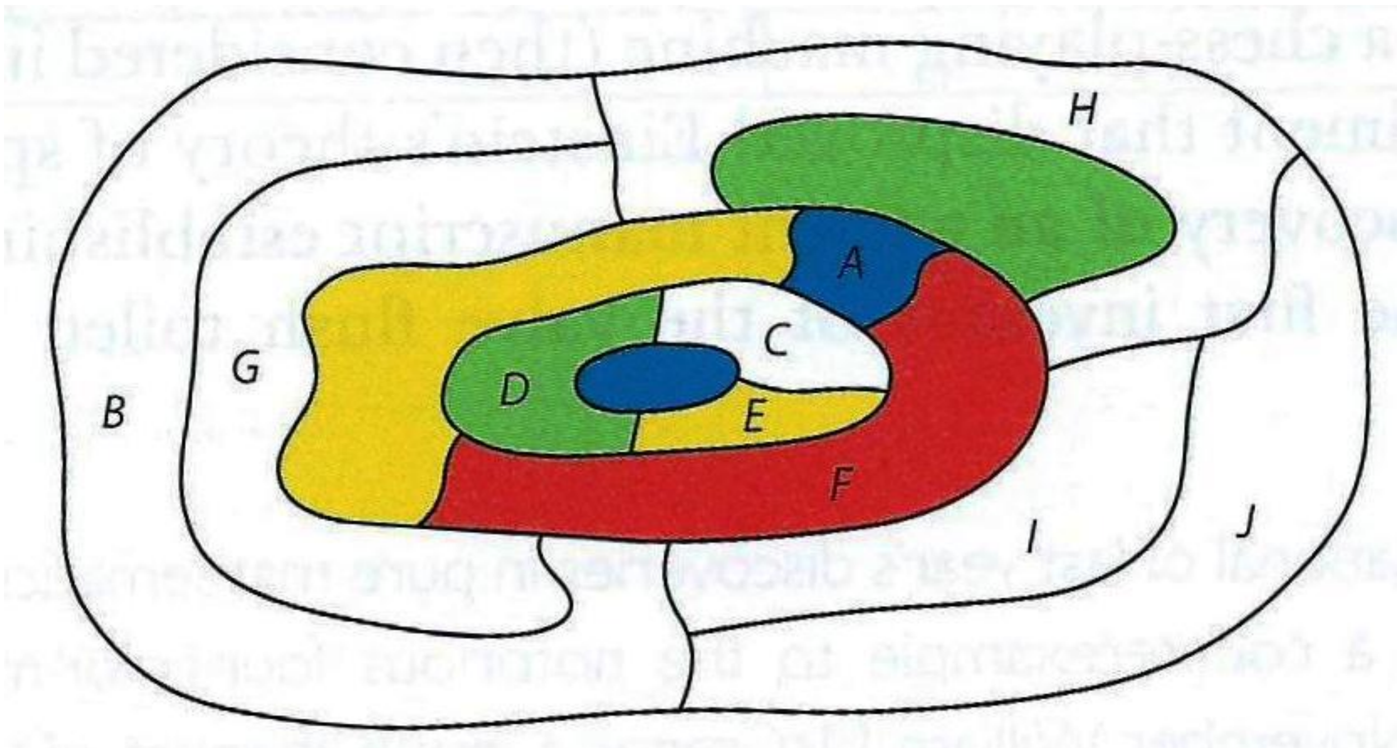
The countries of this map are to be colored red, blue, green, and yellow.

What color is country *B*?



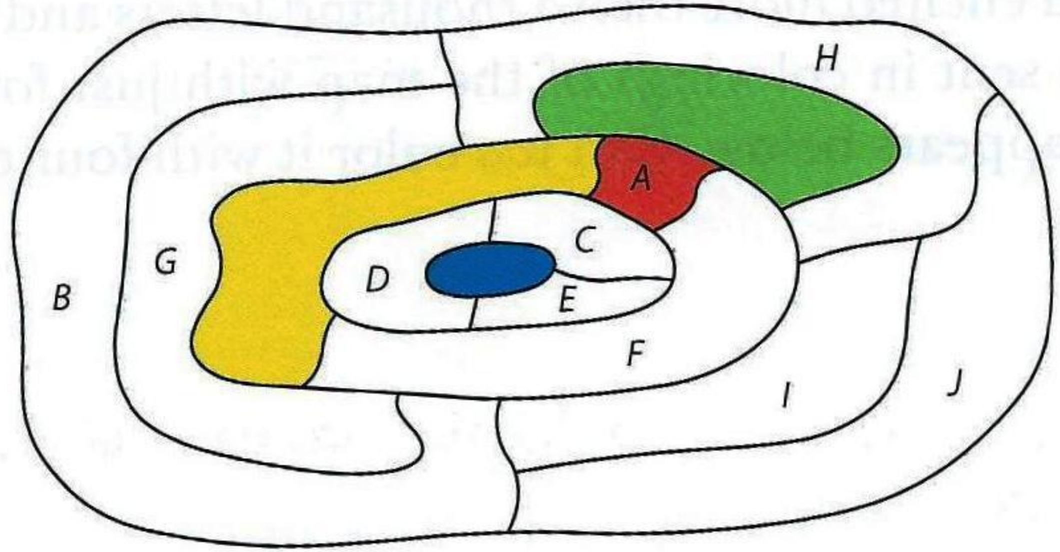
Country *A* must be blue or red

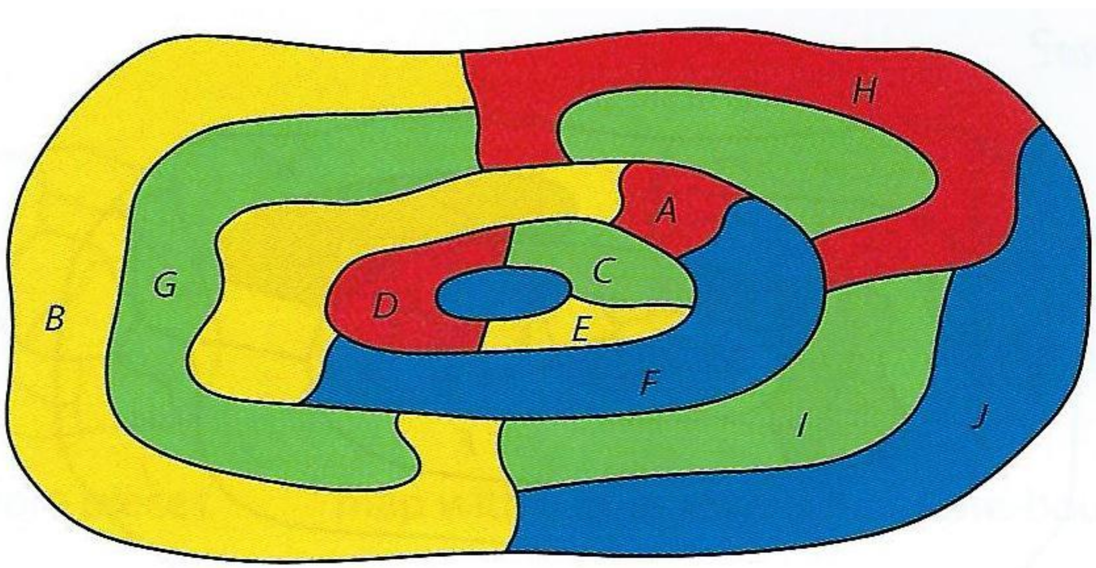
Try blue first: if country *A* is blue . . .



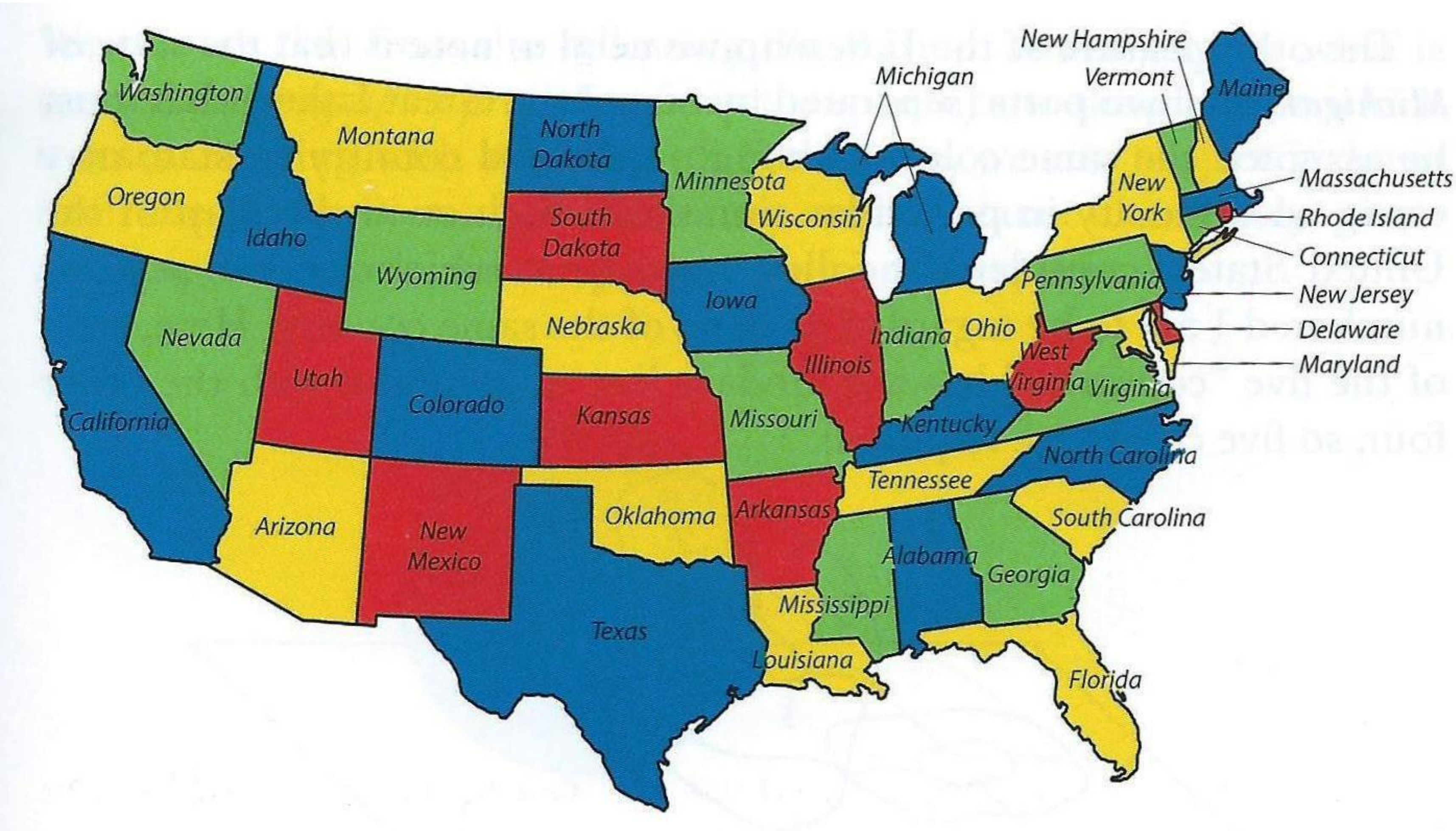
then *F* is red, *D* is green, *E* is yellow
and we can't then color *C*

So country *A* is red, country C is green, . . . , and we can complete the coloring: country *B* is yellow

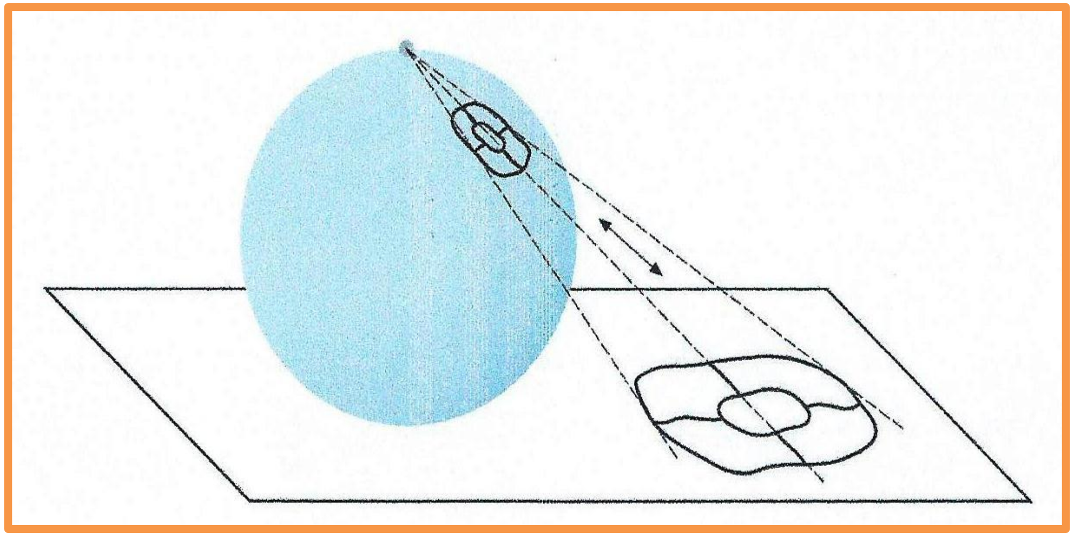




Coloring the USA

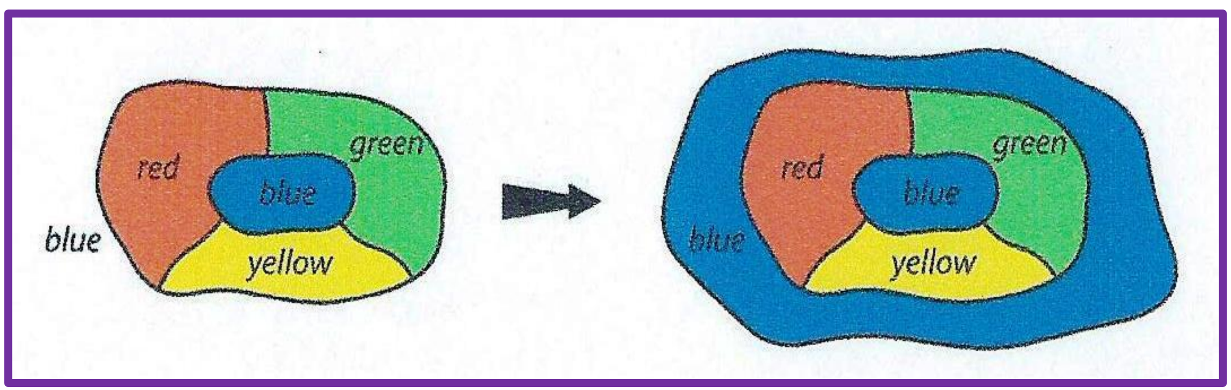


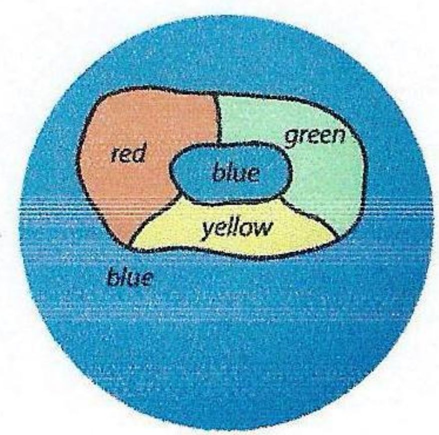
Two observations



The map can be on a plane or a sphere

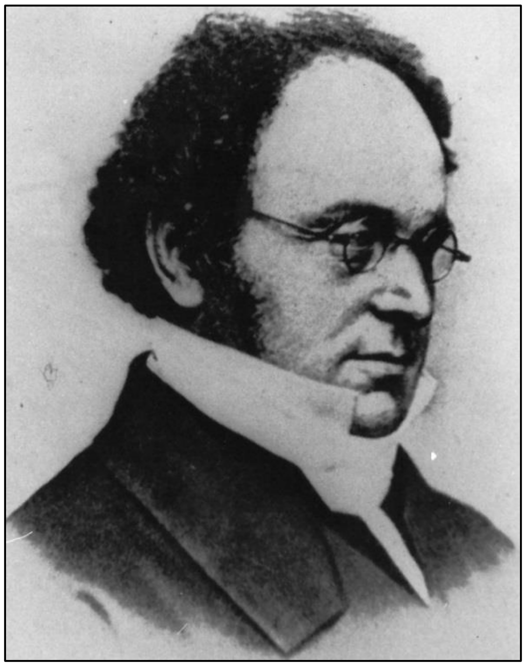
It doesn't matter whether we include the outside region



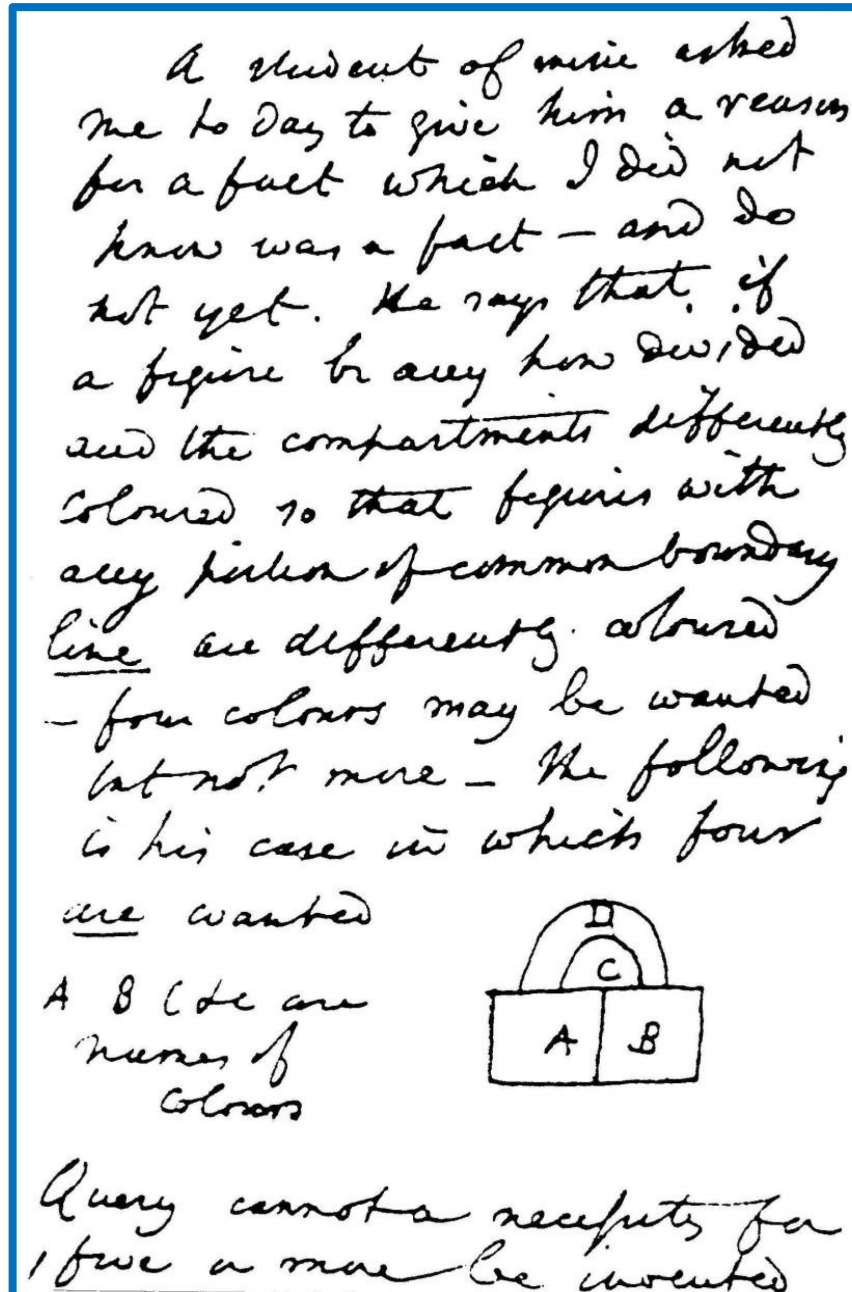


De Morgan’s letter to W. R. Hamilton

23 October 1852



The student was Frederick Guthrie, Francis’s brother, who’d been coloring a map of England



A student of mine asked me to day to give him a reason for a fact which I did not know was a fact — and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured — four colours may be wanted but not more — the following is his case in which four are wanted

A B C D are names of colours

D
C
A
B

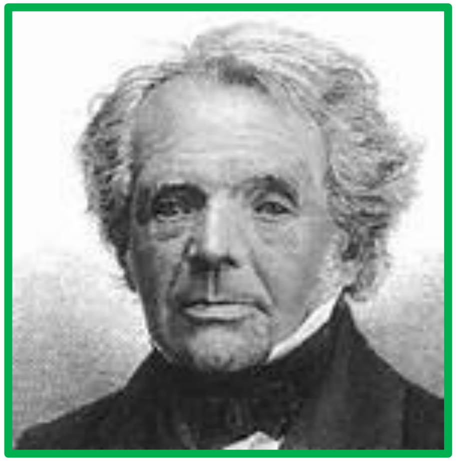
Query cannot a necessity for five or more be invented

The first appearance in print?

F. G. in *The Athenaeum*, June 1854

Tinting Maps.—In tinting maps, it is desirable for the sake of distinctness to use as few colours as possible, and at the same time no two conterminous divisions ought to be tinted the same. Now, I have found by experience that *four* colours are necessary and sufficient for this purpose,—but I cannot prove that this is the case, unless the whole number of divisions does not exceed five. I should like to see (or know where I can find) a general proof of this apparently simple proposition, which I am surprised never to have met with in any mathematical work. F. G.

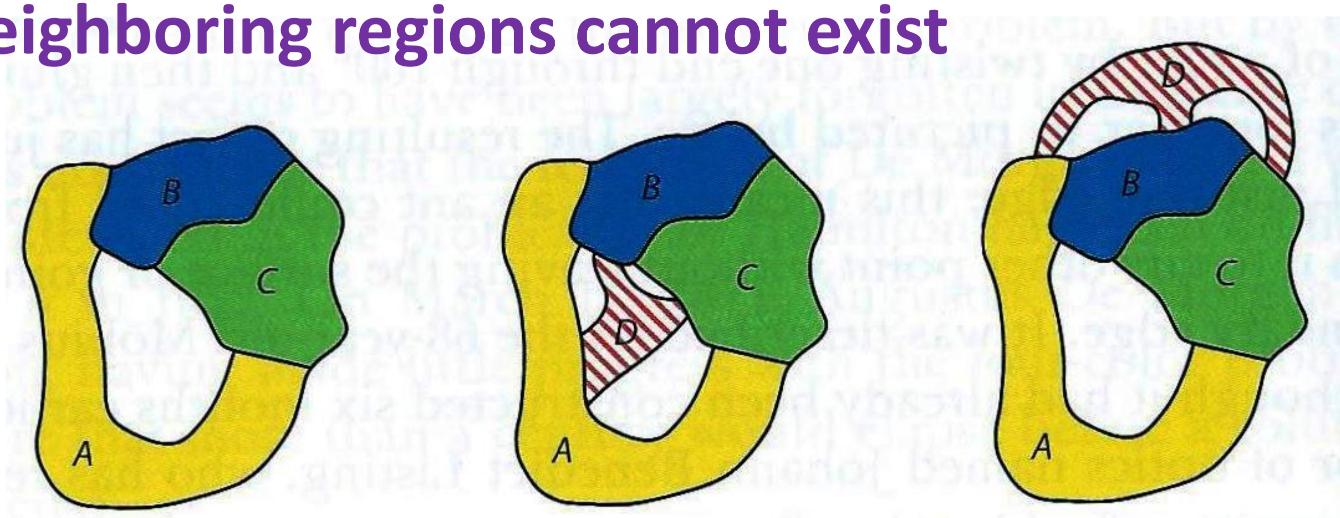
Möbius and the five princes (c.1840)



A king on his death-bed:

'My five sons, divide my land among you, so that each part has a border with each of the others.'

Möbius's problem has no solution:
five neighboring regions cannot exist



Some logic . . .

A solution to Möbius's problem would give us a 5-colored map:

'5 neighboring regions exist' implies that 'the 4-color theorem is false'

and so

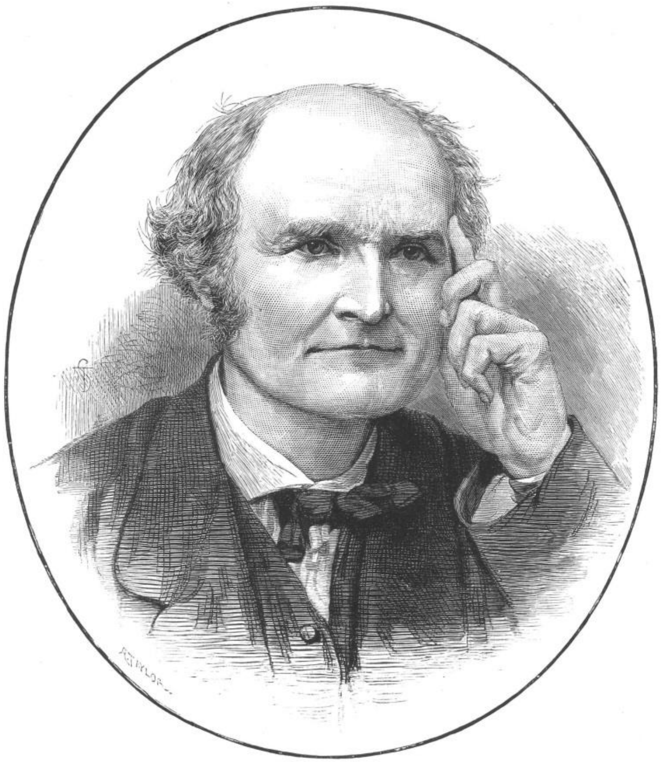
'the 4-color theorem is true' implies that '5 neighboring regions don't exist'

BUT

'5 neighboring regions don't exist' DOESN'T imply that 'the 4-color theorem is true'

So Möbius did NOT originate the 4-color problem

Arthur Cayley revives the problem

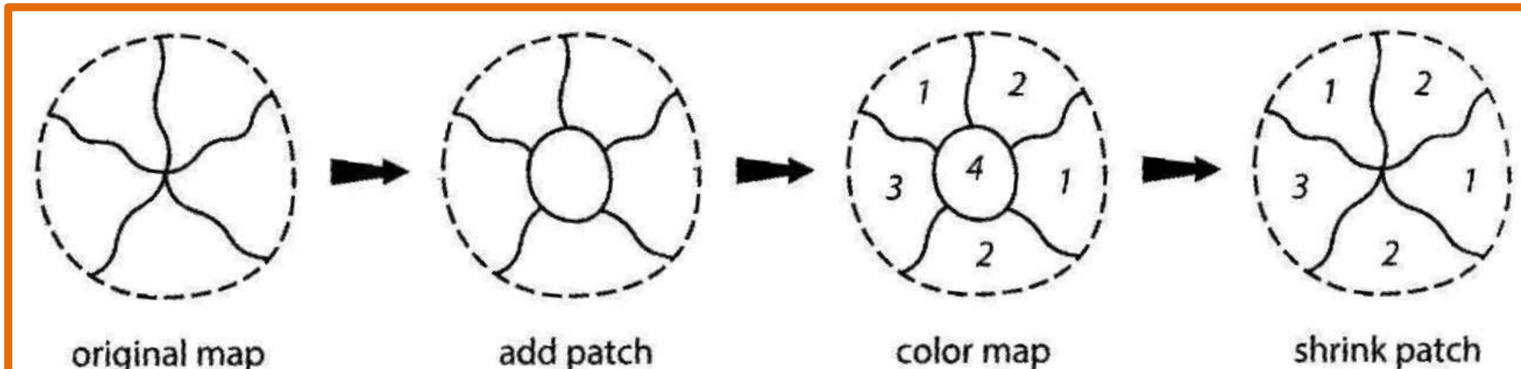


13 June 1878

London Mathematical Society

Has the problem been solved?

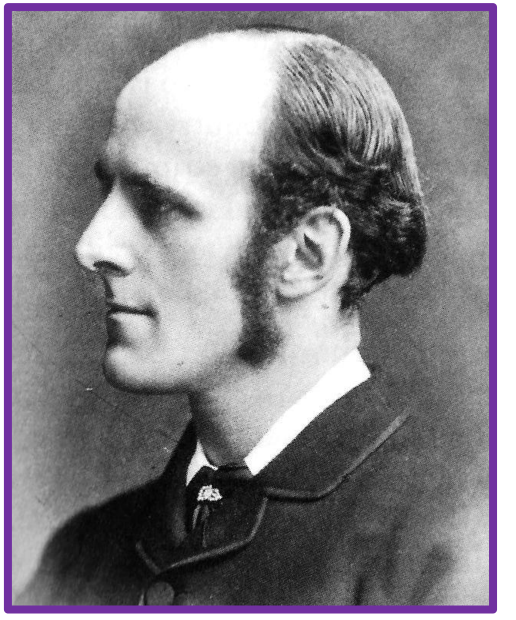
1879: short note: we need consider only 'cubic' maps (3 countries at each point)



A. B. Kempe 'proves' the theorem

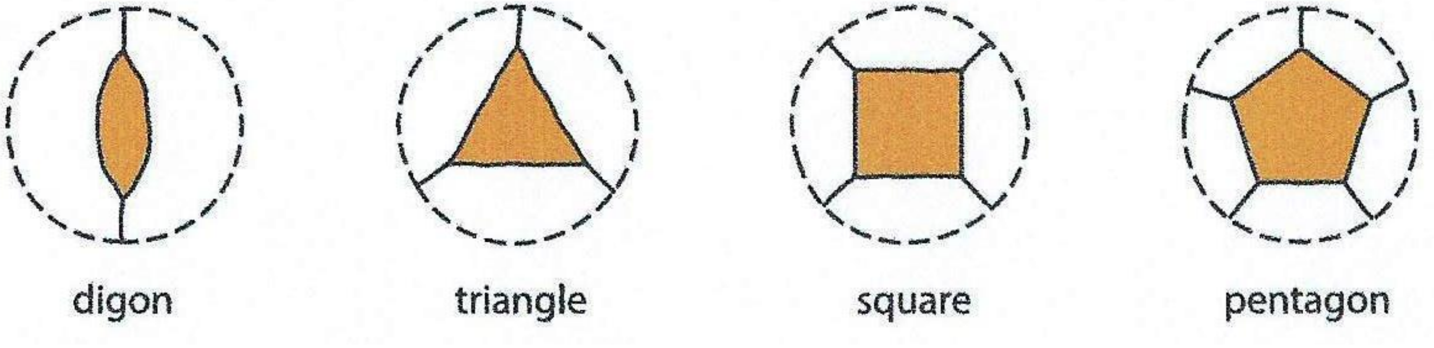
American Journal of Mathematics, 1879

'On the geographical problem of the four colours'



From Euler's polyhedron formula:

Every map contains a digon, triangle, square, or pentagon



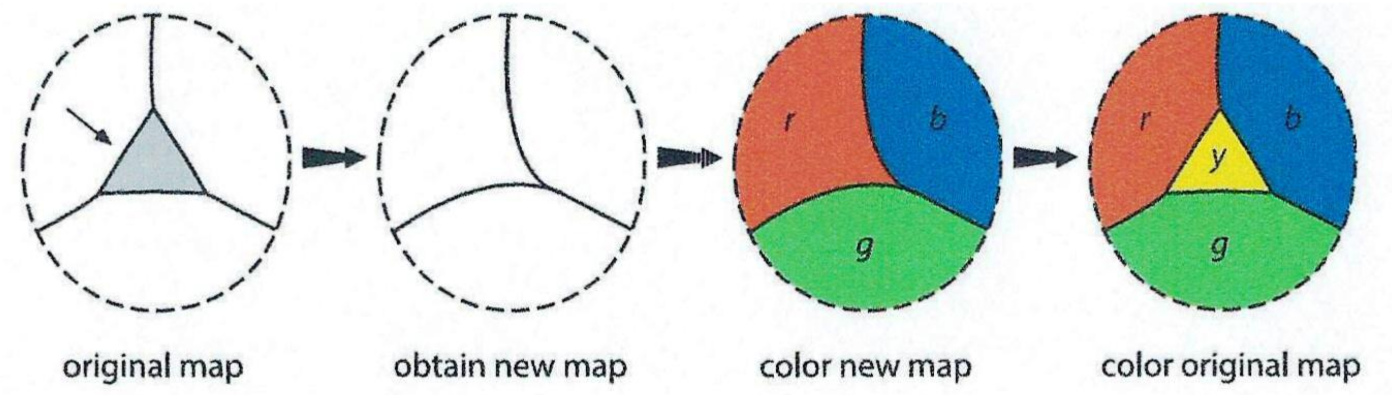
Kempe's proof 1: digon or triangle

Every map can be 4-colored

Assume not, and let M be a map with the smallest number of countries that cannot be 4-colored.

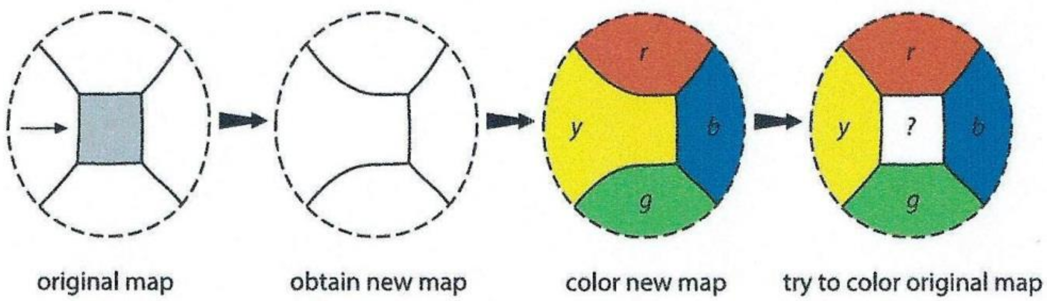
If M contains a digon or triangle T, remove it, 4-colour the resulting map, reinstate T, and color it with any spare color.

This gives a 4-coloring for M: contradiction

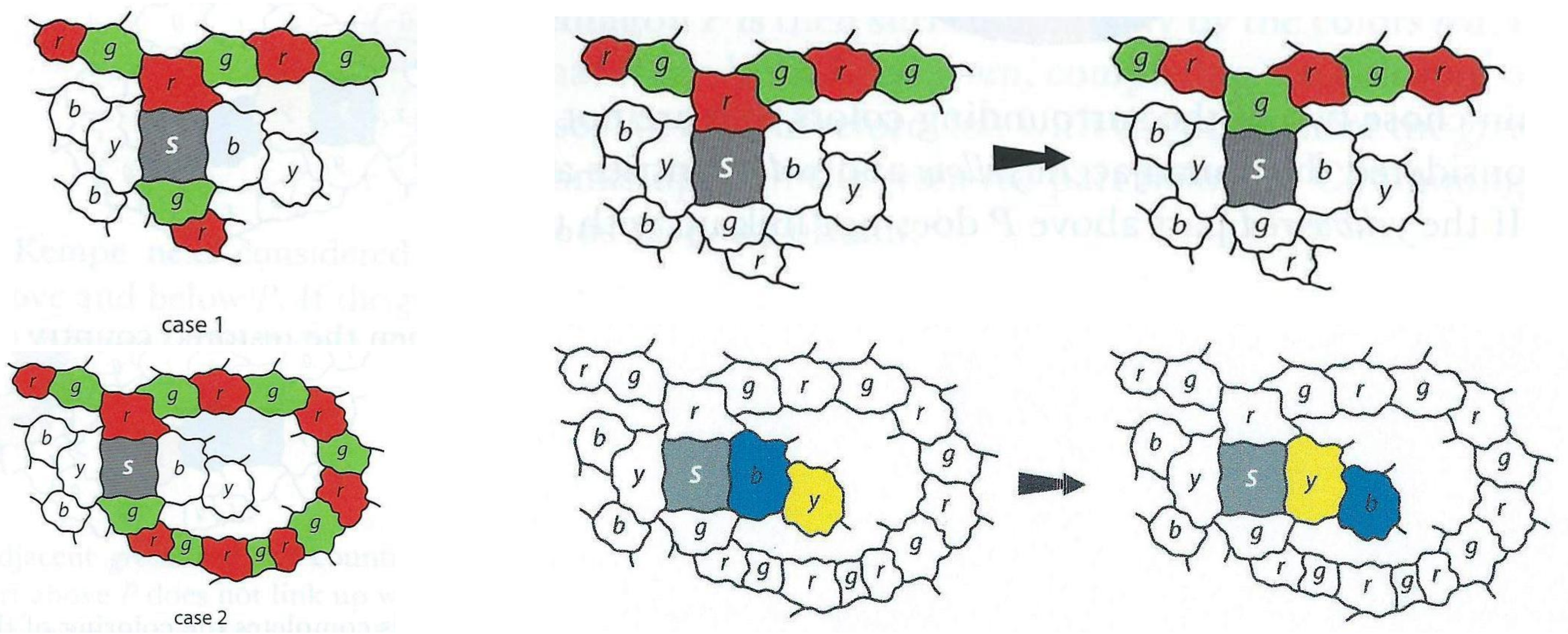


Kempe's proof 2: square

If the map M contains a square S, try to proceed as before:

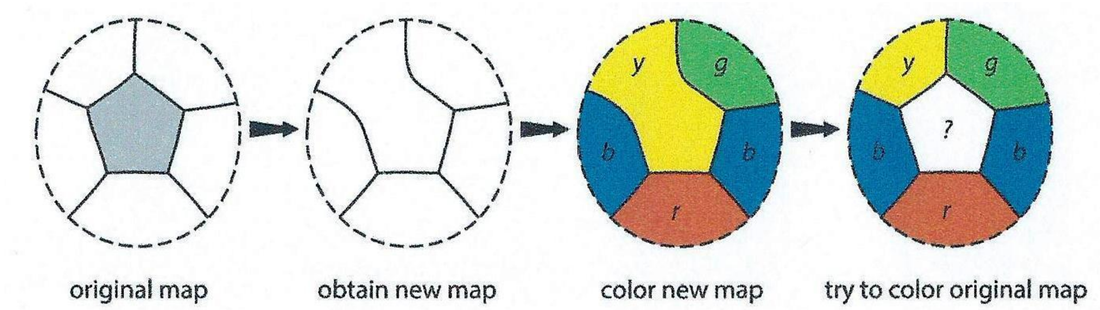


Are the red and green countries joined? Two cases:

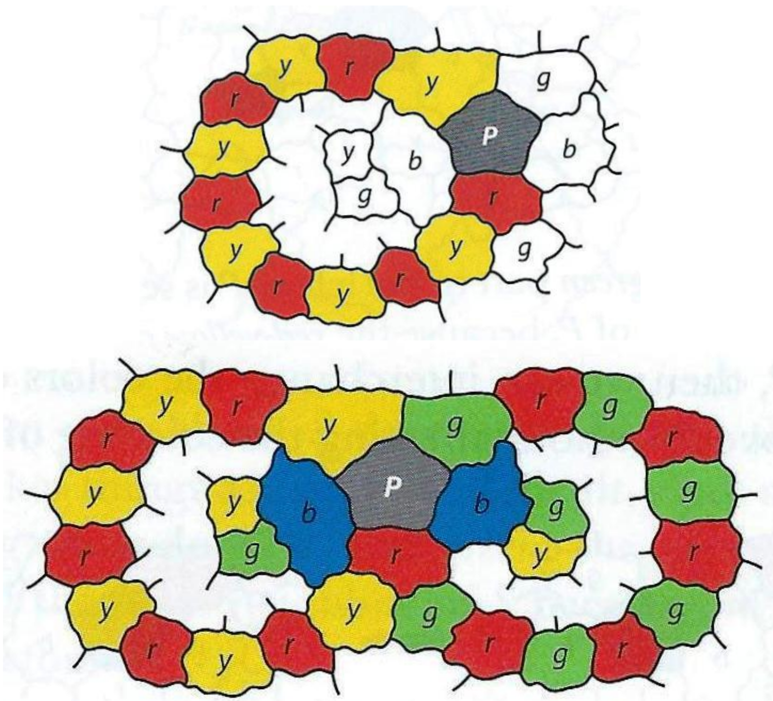


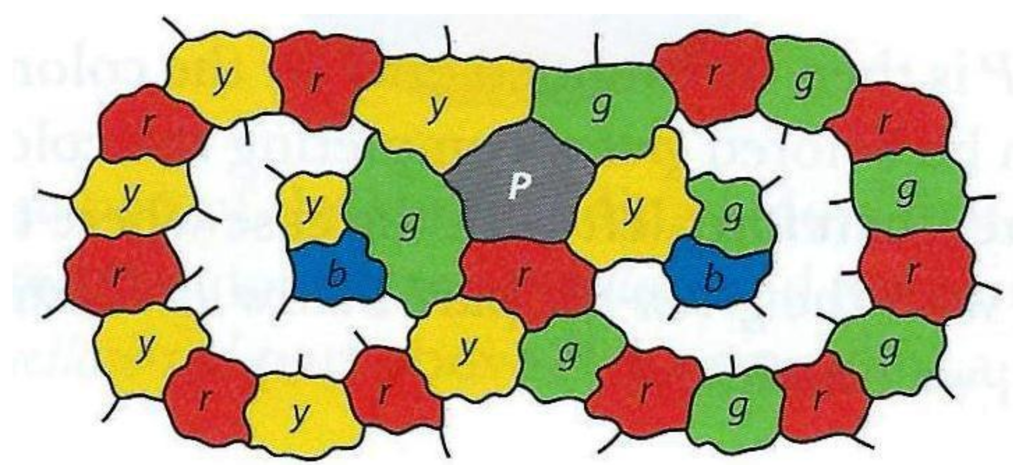
Kempe's proof 3: pentagon

If the map M contains a pentagon P:

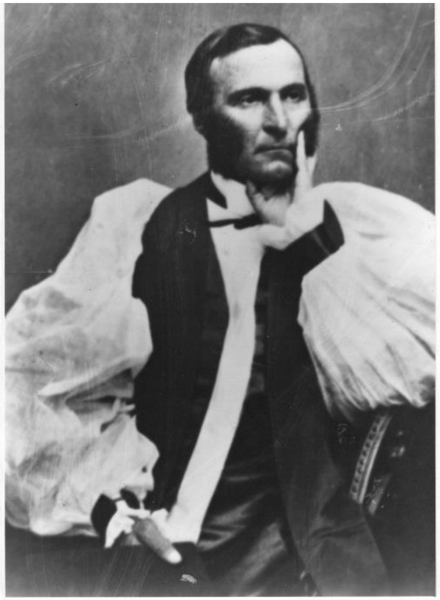


Carry out TWO 'Kempe interchanges' of color:





The problem becomes popular . . .



Lewis Carroll turned the problem into a game for two people . . .

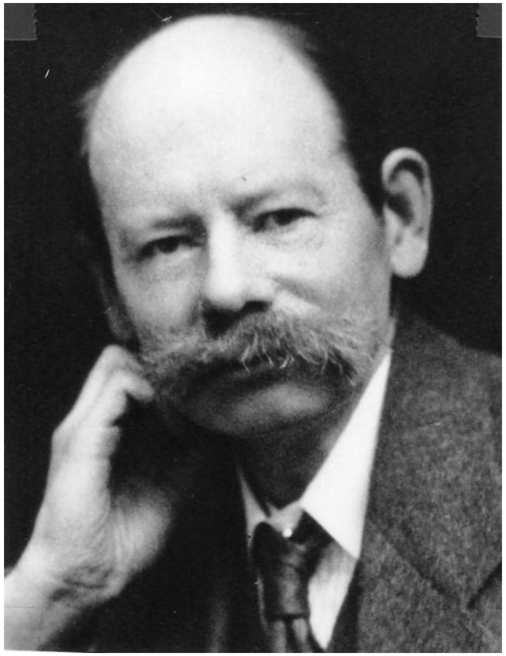
1886: J. M. Wilson, Headmaster of Clifton College, set it as a challenge problem for the school

1887: . . . and sent it to the *Journal of Education*

. . . who in 1889 published a 'solution' by Frederick Temple, Bishop of London

Percy Heawood's 'bombshell'

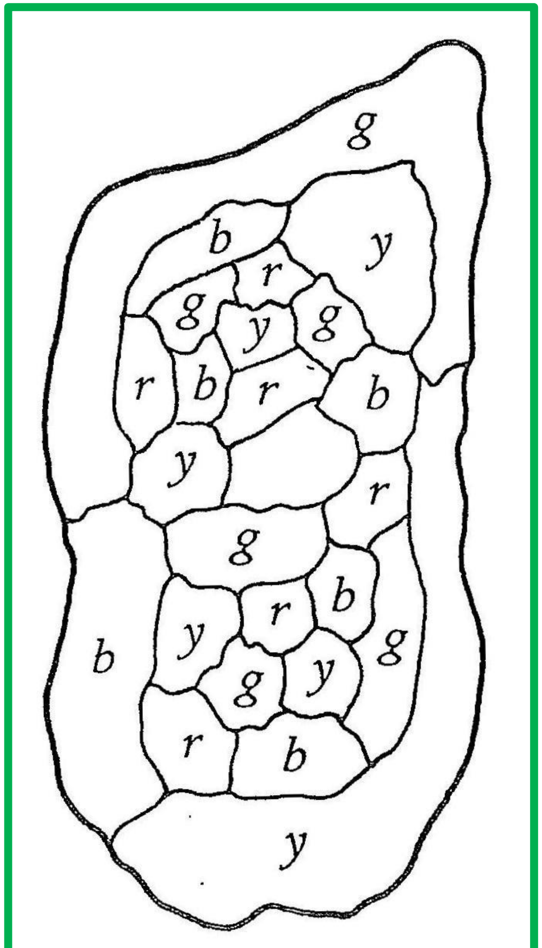
1890: 'Map-colour theorem'

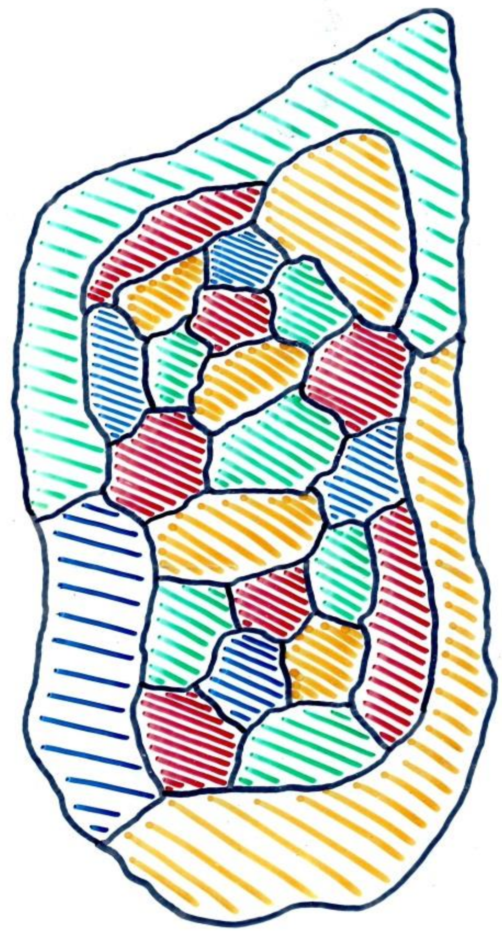


- pointed out the error in Kempe's proof
- salvaged enough from it to prove the 5-color theorem
- generalized the problem from the sphere to other surfaces

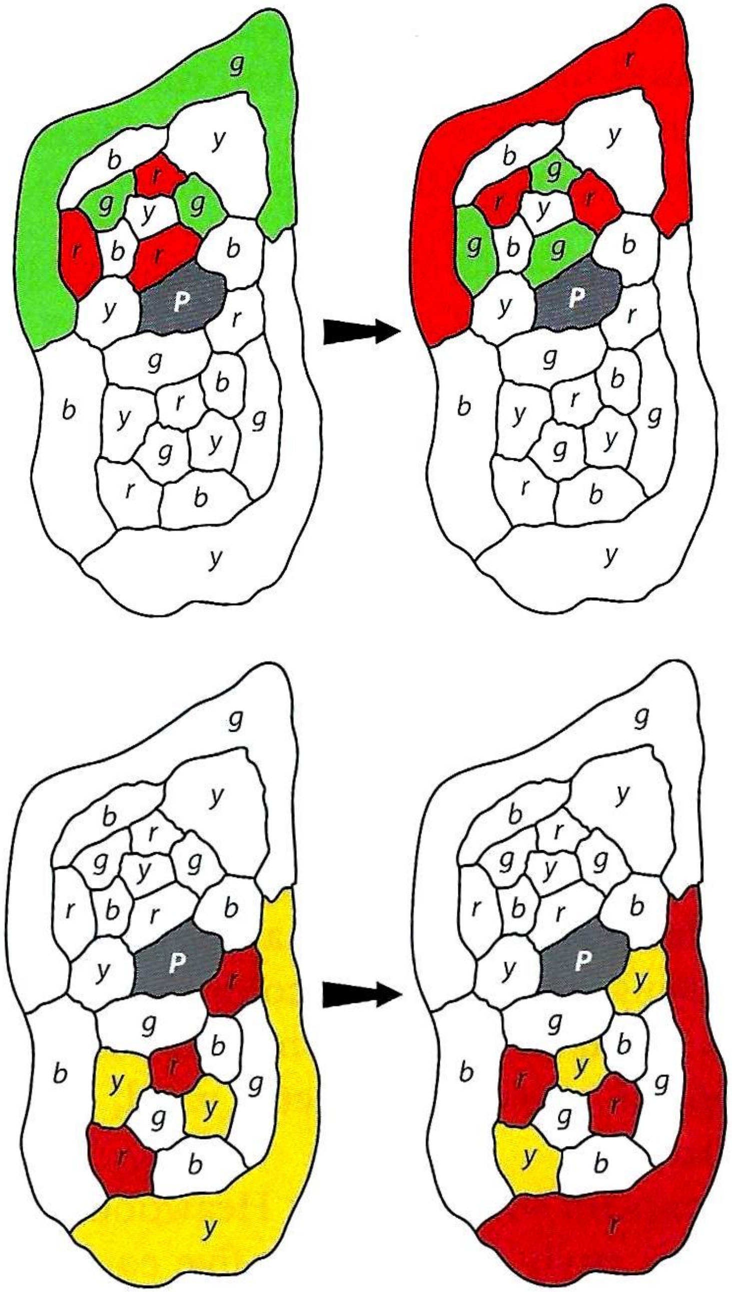
Heawood’s example 1

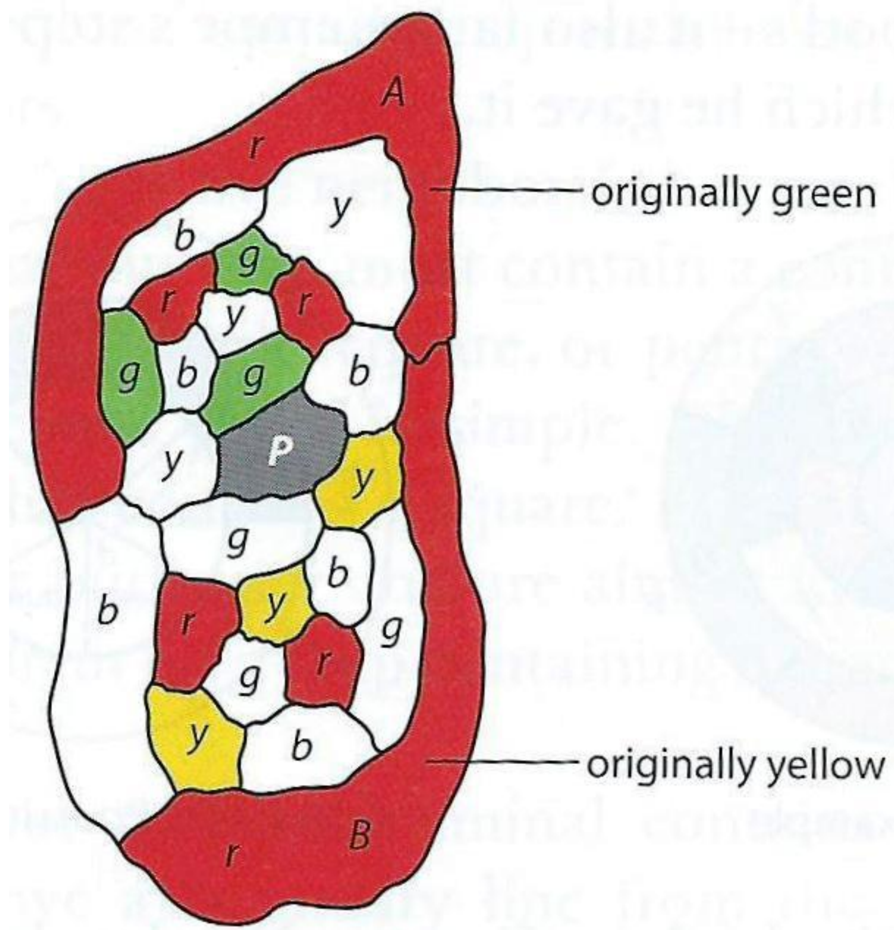
You cannot do two Kempe interchanges at once . . .





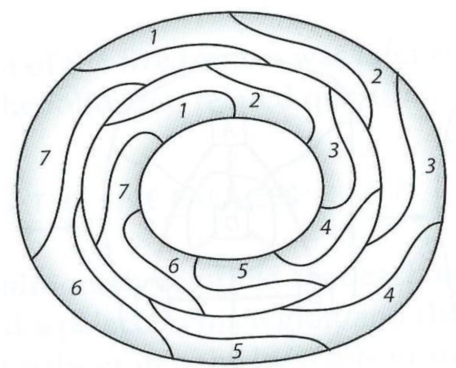
Heawood's example 2





Maps on other surfaces

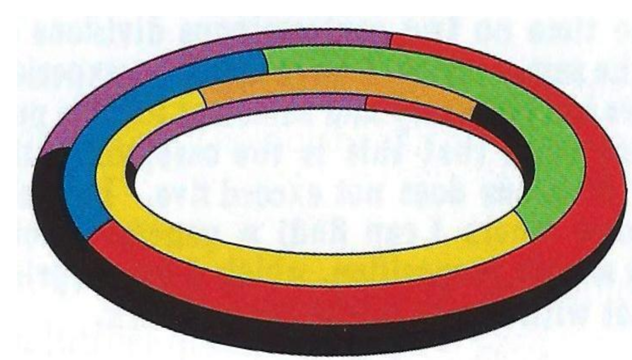
The four-color problem concerns maps on a plane or sphere . . . but what about other surfaces?



TORUS

7 colors suffice . . .

and may be necessary



HEAWOOD CONJECTURE

For a surface with h holes ($h \geq 1$)

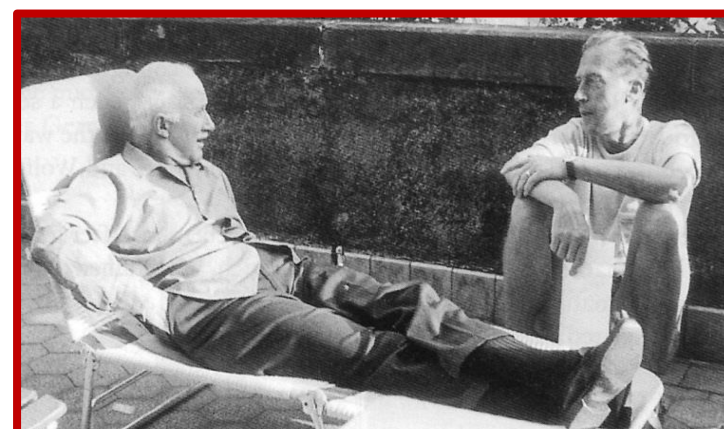
$[\frac{1}{2}(7 + \sqrt{1 + 48h})]$ colors suffice

$h = 1$: $[\frac{1}{2}(7 + \sqrt{49})] = 7$

$h = 2$: $[\frac{1}{2}(7 + \sqrt{97})] = 8$

But do we need this number of colors?

YES: G. Ringel & J. W. T. Youngs (1968)



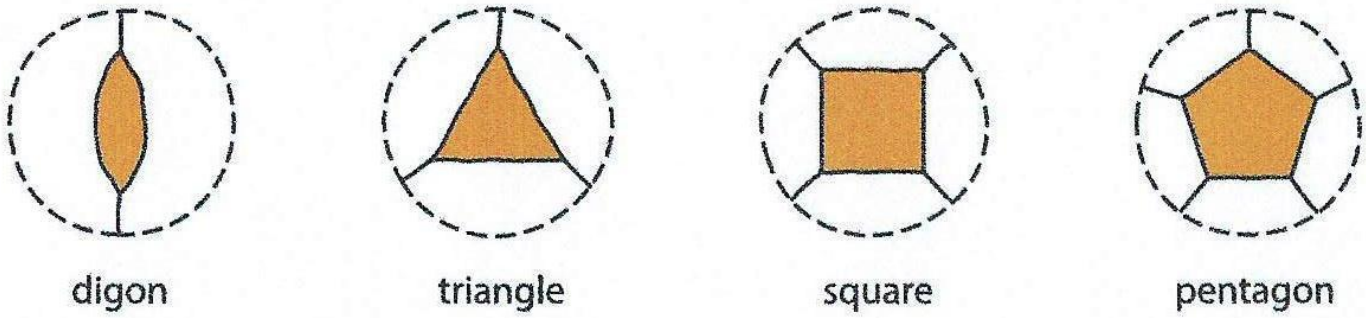
Two main ideas

A **configuration** is a collection of countries in a map.

We shall be concerned with

- unavoidable sets of configurations
- reducible configurations

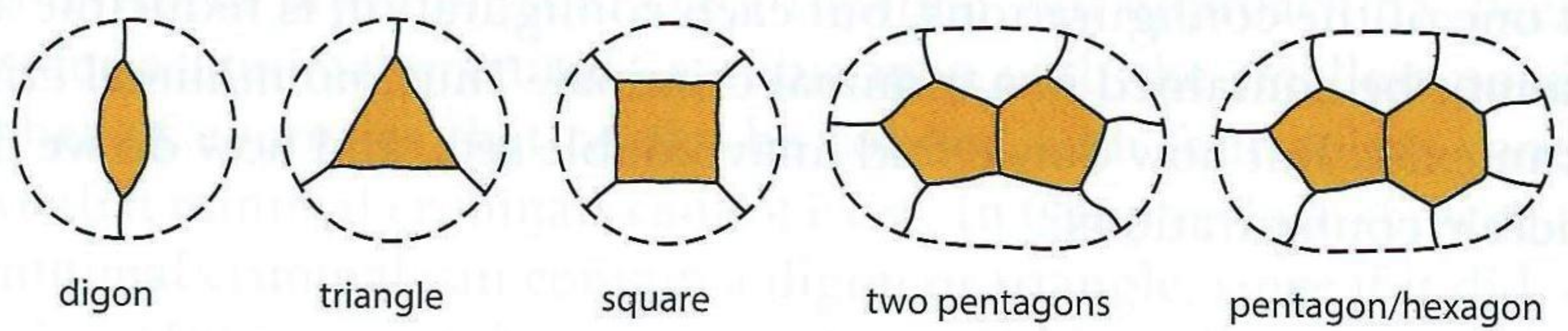
Unavoidable sets



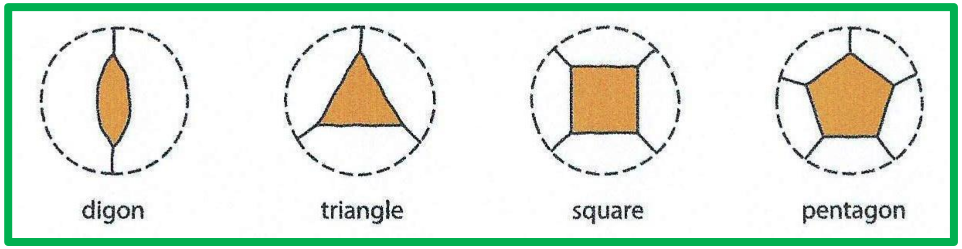
is an unavoidable set:

every map contains at least one of them

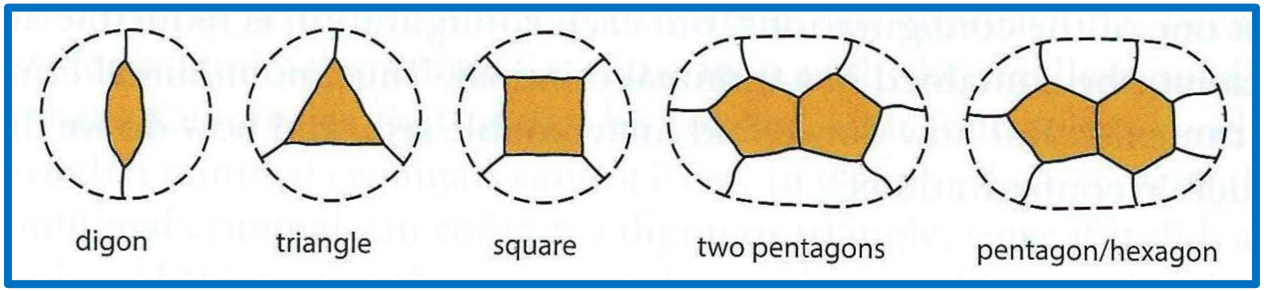
and so is the following set of Wernicke (1904):



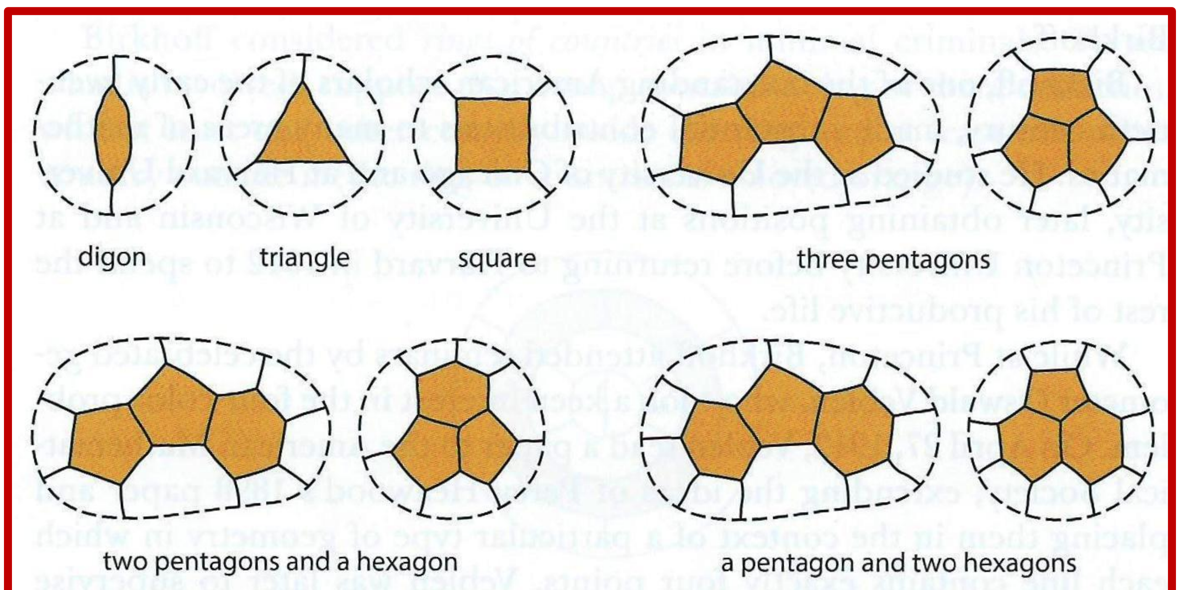
Unavoidable sets



Kempe 1879



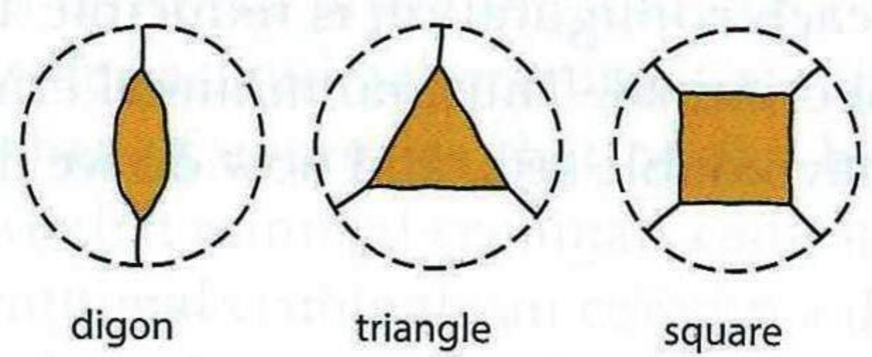
Wernicke 1904



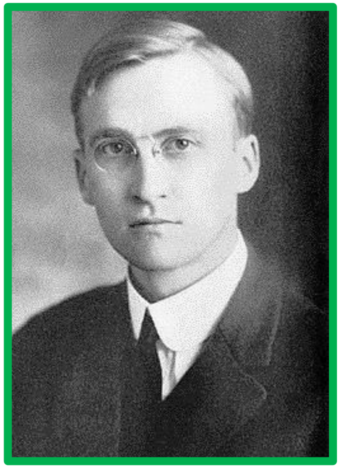
P. Franklin 1922:
so the four-color theorem is true for all maps with up to 25 countries.

Later sets found by H. Lebesgue (1940).

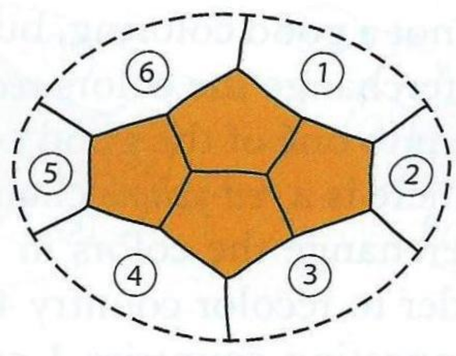
Reducible configurations



Each of these configurations is 'reducible':
any coloring of the rest of the map
can be extended to include them

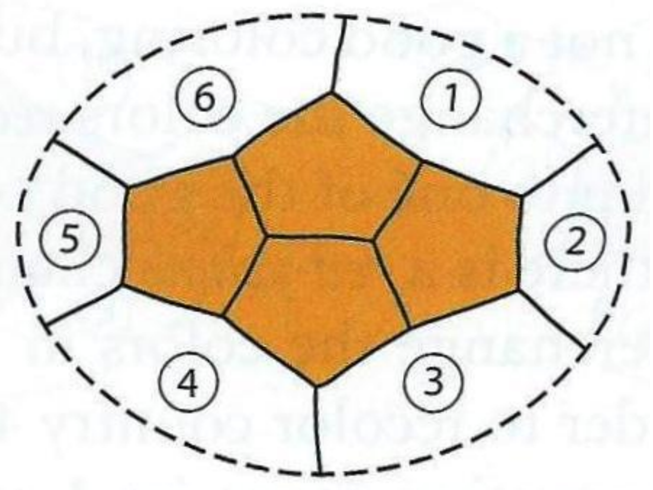


So is the 'Birkhoff diamond' (1913)



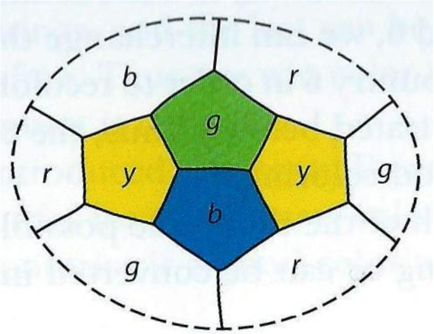
Testing for reducibility

Color the countries 1–6 in all 31 possible ways:



Birkhoff diamond

	rgrgrg	*rgrbrg**	*rgrbgy**	*rgbrgy*	*rgbryb*	*rgbgbg**	*rgbyrg*	*rgbygy**
→	*rgrgrb**	*rgrbrb*	*rgrbyg**	*rgbrbg**	*rgbgrg**	*rgbgby*	*rgbyrb*	*rgbybg**
	rgrgbg	*rgrbry*	*rgrbyb**	*rgbrby*	*rgbgrb**	*rgbgyg*	*rgbyry**	*rgbyby**
	*rgrgby**	*rgrbgb**	*rgbrgb*	*rgbryg*	*rgbgry**	*rgbgyb*	*rgbygb*	

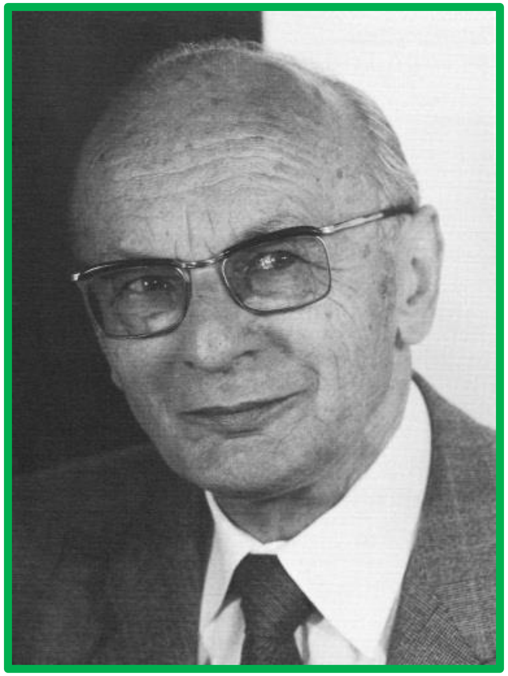


rgrgrb **extends directly:**

In fact, ALL can be done directly or via Kempe interchanges of color

Enter Heinrich Heesch (1906-95)

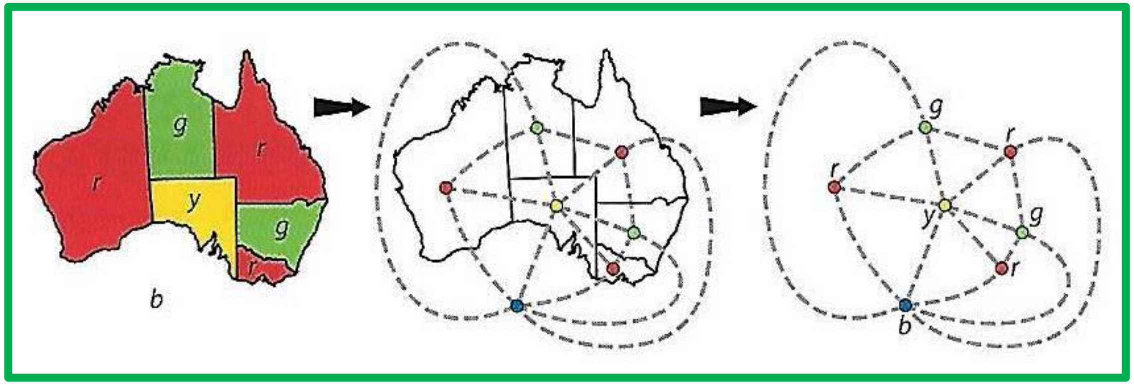
- In 1932 he solved Hilbert's Problem 18 on tilings of the plane.
- He invented the 'method of discharging' for unavoidable sets, and found thousands of reducible configurations.
- He estimated that 10,000 configurations might need to be tested, up to 'ring-size' 18.
- He gave lectures on the 4-color problem at the University of Kiel, attended by Haken.

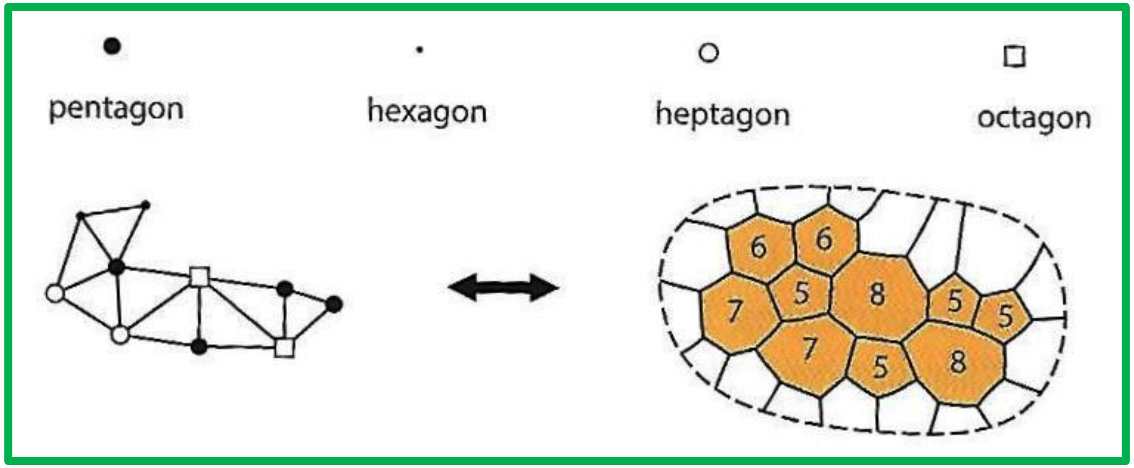


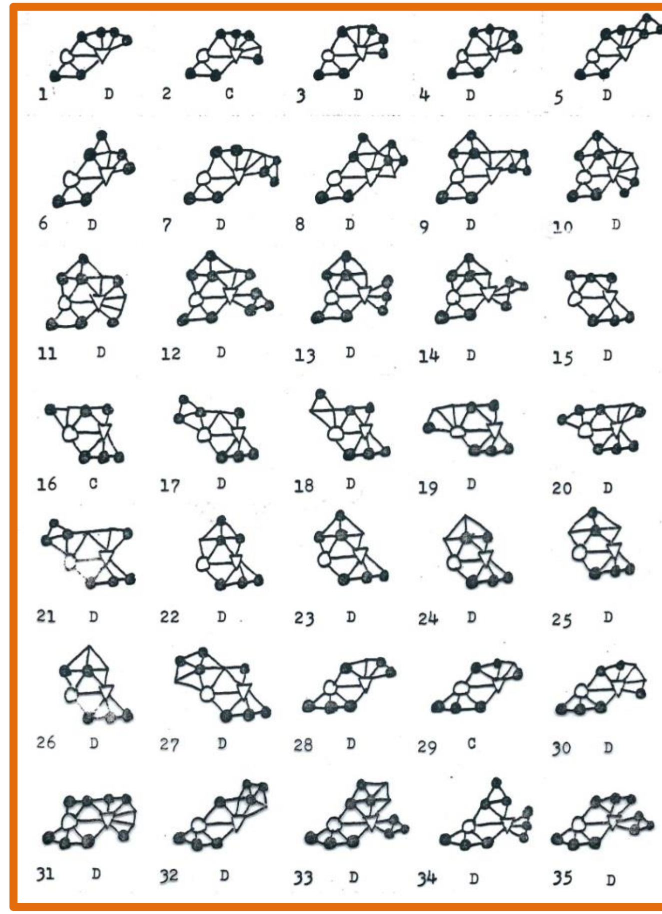
To solve the four color problem, find an unavoidable set of reducible configurations

Every map must contain at least one of them, and whichever it is, any coloring of the rest of the map can be extended to it.

Maps versus graphs

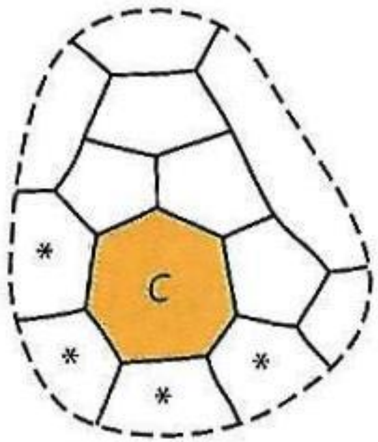




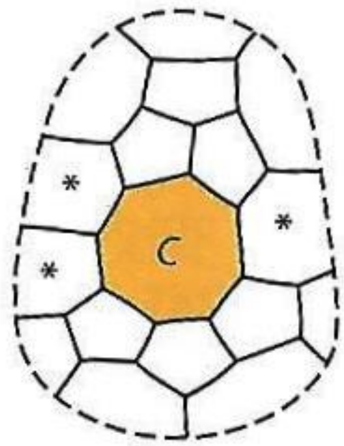


Appel & Haken, 1977

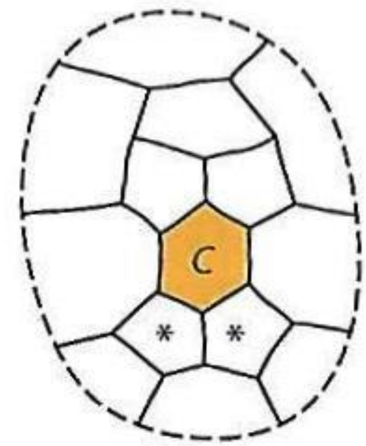
Three obstacles to reducibility



4–legger country



3–legger articulation country



hanging 5–5 pair

If any of these appears in a configuration, then it's likely not to be reducible.

Enter Wolfgang Haken

Three problems:

The knot problem

(solved completely in 1954)

The Poincaré conjecture

(almost solved)

The four-color problem

(solved with Ken Appel in 1976)

"Mathematicians usually know when they have gotten too deep into the forest to proceed any further. That is the time Haken takes out his penknife and cuts down the trees one at a time."

Enter Kenneth Appel

Heesch, Haken, and others were already using computers to test reducibility, with a certain amount of success. But the problem was quickly becoming too big to handle, possibly with thousands of large configurations, each taking many hours of computer time.

Haken, in a lecture at the University of Illinois

"The computer experts have told me that it is not possible to go on like that. But right now I'm quitting. I consider this to be the point to which and not beyond one can go without a computer."

In the audience was Appel, an experienced computer programmer

"I don't know of anything involving computers that can't be done: some things just take longer than others. Why don't we take a shot at it?"

1976
Kenneth Appel
& Wolfgang Haken
(Univ. of Illinois)

Every planar map is four colorable
(with John Koch)

They solved the problem by finding an unavoidable set of 1936 (later 1482) reducible configurations.

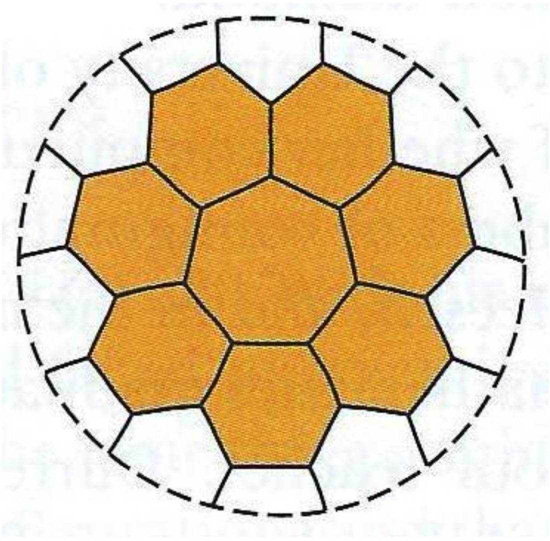
The Appel-Haken approach

They developed a 'discharging method' that yields an unavoidable set of 'likely-to-be-reducible' configurations.

They then used a computer to check whether these configurations are actually reducible: if not, modify the unavoidable set.

They had to go up to 'ring-size' 14.

(199,291 colorings)



The proof is widely acclaimed

第3種郵便物認可

変わる証明の概念

電算機利用は不可欠に

「四色問題」解決の意義

どんな地図でも4色で塗り分けられる

四色問題の証明は、数学の専門家だけではなく、一般の数学ファンにとっても関心の深い問題だった。「平面または球面上のすべての地図は四色で塗り分けられる。これを証明せよ」という問題は、それほど難しい数式を使わなくても取り組みやすそうに見えるし、素人受けするテーマだ。百三十年前、数学者にこの問題を持ち込んで証明をたのんだのは、地図印刷業者が経験的に知っていたことを小耳にはさんだ大学生だった、といわれている。最初は、数学者たちには簡単に解けそうに思えた。が、やってみると難しかった。

このように、ある定理の真偽が、肯定も否定もされずに何百年も未解決でいることは数学上の大問題であるだけでなく「数学者の精神衛生上の大問題」となる。なぜなら、一般の数学ファンがこれでもかこれでもかと"証明成功"の論文を数学者たちに持ち込んで点検を迫り、悩ますからだ。

「たとえば、角の三等分の問題。これは定規とコンパスではできない。できないことを数学者が証明ずみだから、どんな珍証明を持ち込まれても、ただちにクズカゴへ捨ててしまう。しかし、四色問題は、そうはいかなかった。大多数の数学者は、四色問題を証明したという数学ファンの論文を、自責の念を持ちつつ、クズカゴへ捨てていたのです」と矢野健太郎東工大名誉教授はいう。数学者の悩みの種は、まだかなり残されている。「中の島の両側にかかった七つの橋を、すべて一度ずつかならず渡る方法はあるか」という有名な「ケーニヒスベルグの七つの橋の問題」など、数学ファンの興味をそそるものだ。

イリノイ大学の二人の数学教授が証明した方法は、数学界に新しい論争を呼び起こしている。紙と鉛筆と人間の理性だけでスマートに解くのではなく、電子計算機を動員して、膨大な計算をした結果はじめて得られた証明が果たして、本当の証明といえるかどうかという議論だ。一松信京大数理解析研究所教授は、電子計算機の功罪を次のようにいう。

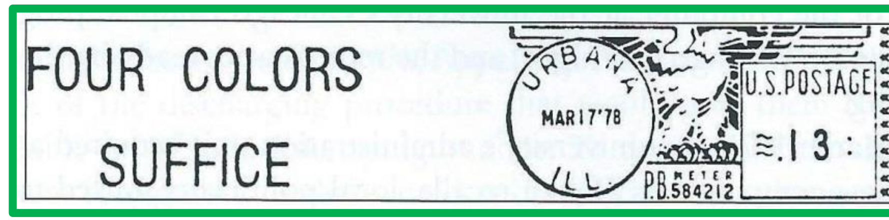
まず、難しそうな問題を電算機にやらせて、証明できたとする。証明できたとわかっていれば、数学者は、電子計算機なしでいずれはスマートな証明法が見つかる、とファイトを燃やせる。「こうして、最初は人間の理性ではとてもダメだと思われていた有限群をスマートに再証明した実例もかなりある」という。

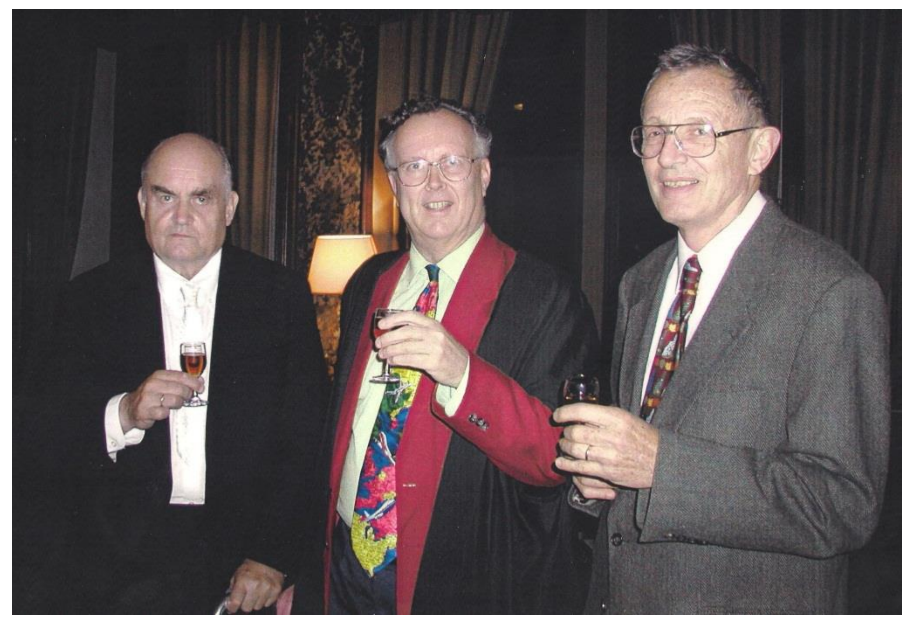
しかし、数学的証明とは、誤りがまったくないものでなければならない。計算機は機械だから、どこかが故障して誤った計算をしているかも知れない。一つの計算機で、その命題を肯定する証明が完了しても、別の計算機は、それを否定する証明を出すかもしれない。このとき、どちらの機械が誤っているのか、第三の電算機が判定者として乗り出してきて、どちらか一方に軍配を上げたとしても、二対一になったに過ぎない。多数決で数学の定理が証明できるとは、伝統的な数学者は考えていない。四月にピッツバーグのカーネギーメロン大学で開かれたコンピューター数学のシンポジウムでも、この点が大議論になった。

にもかかわらず、数学の定理を電子計算機で証明するという新しい方法は、将来の数学にとって必要不可欠な方法となる、とみる数学者はふえている。数学者自身「証明」の定義と概念を変えざるを得ない事態に来ているのだ。なぜならスマートな証明ができるような問題は、ほとんどすべて証明ずみとなり、あとは電算機に頼るしかないような複雑な問題しか残っていないと予想されるからだ。

しかし、現在、数学者は電子計算機を十分には使えない状況だ。数学者以外の科学者の使用率が圧倒的に高いうえに、巨額の費用と時間がかかる。電算機の使用を最初から敬遠せざるを得ない数学者も多い。イリノイ大学の二教授の成功は、たまたま新設の電算機を千二百時間もぶっ続けに使える幸運に恵まれたからだ、という見方もある。

「定理の発見者、証明者、証明年月日のほかに、数学の論文にはもう一つのデータが不可欠となった。その定理を証明するのにいくらかかったかを明記する必要があるわけです」と佐藤大八郎カナダ・レジャイナ大学教授は、ため息をついている。（西村　幹夫記者）





Aftermath

The ‘computer proof’ was greeted with suspicion, derision and dismay – and raised philosophical issues. Is a ‘proof’ really a proof if you can’t check it by hand?

Some minor errors were found in Appel and Haken’s proof, and corrected.

Using the same approach, N. Robertson, P. Seymour, D. Sanders, and R. Thomas obtained a more systematic proof in 1994, involving about 600 configurations.

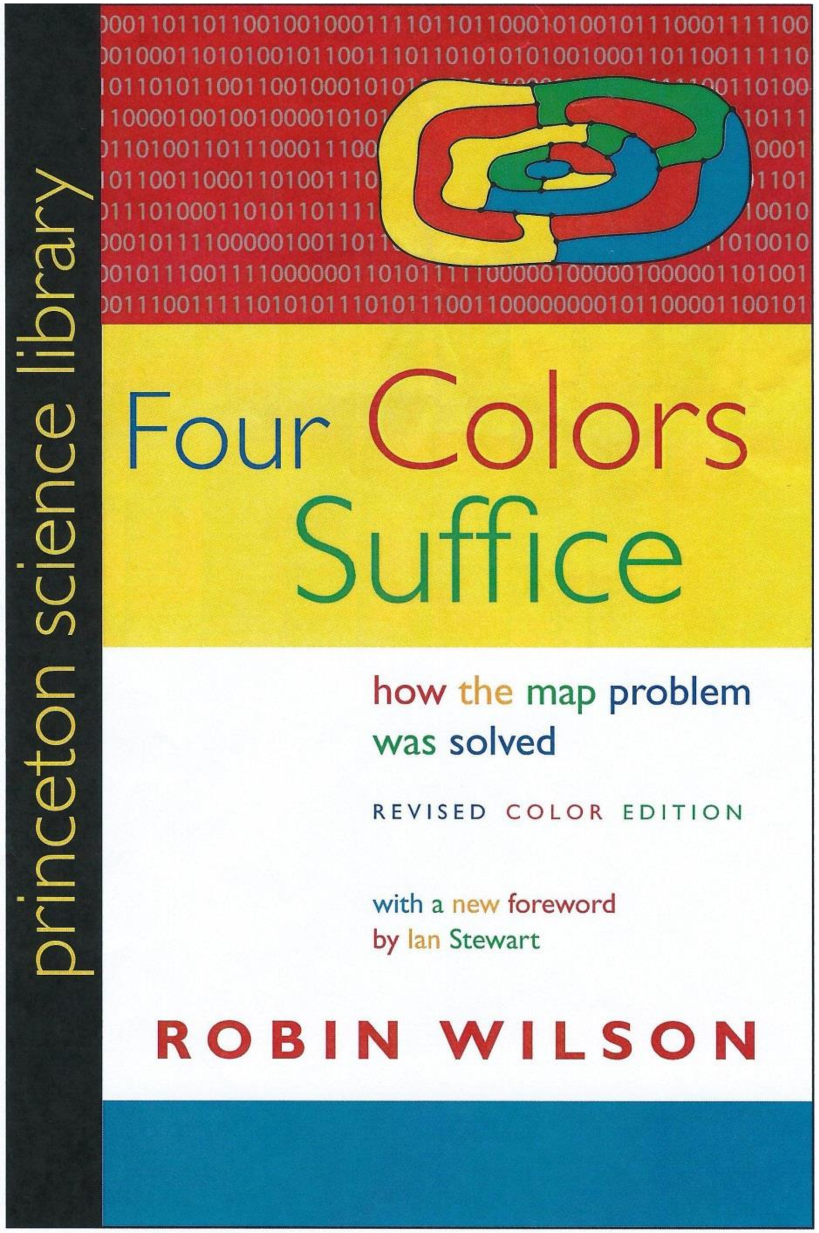
In 2004 G. Gonthier produced a fully machine-checked proof of the four-color theorem (a formal machine verification of Robertson *et al.*’s proof).

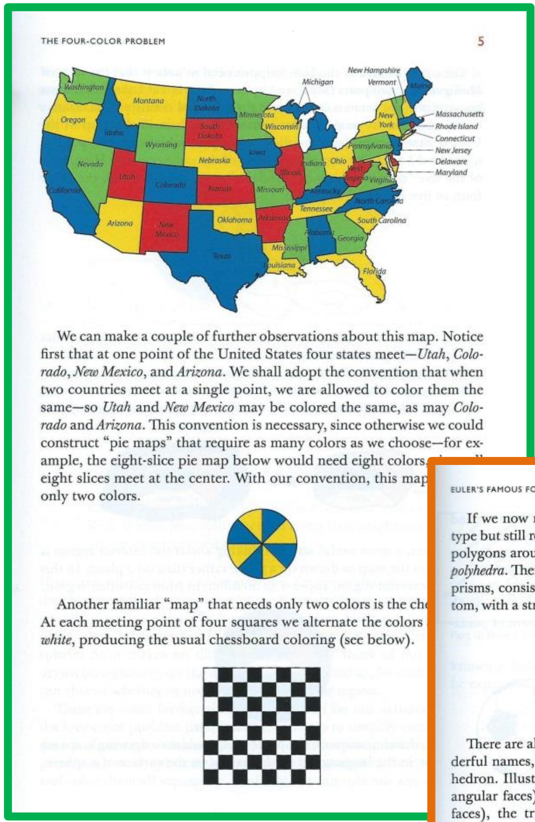
The story is not finished

Many new lines of research have been stimulated by the four-color theorem, and there are several conjectures of which it is but a special case.

In 1978 W. T. Tutte wrote:

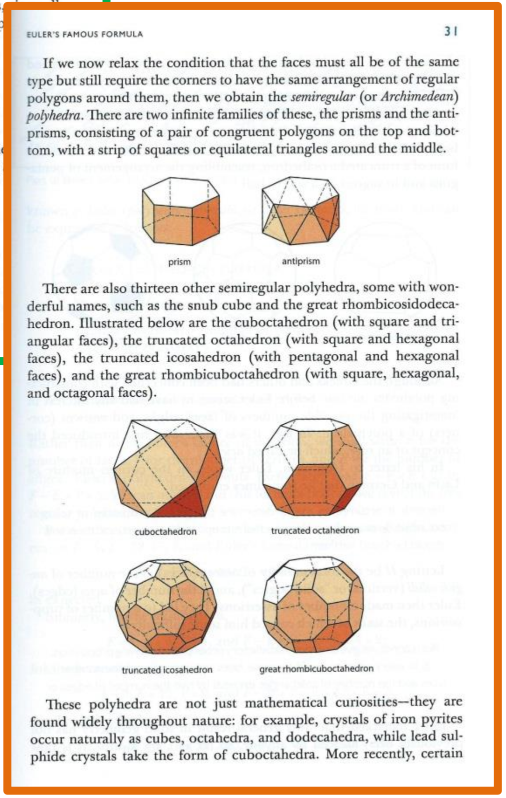
**The Four Colour Theorem
is the tip of the iceberg,
the thin end of the wedge
and the first cuckoo of Spring**




THE FOUR-COLOR PROBLEM 5

We can make a couple of further observations about this map. Notice first that at one point of the United States four states meet—*Utah*, *Colorado*, *New Mexico*, and *Arizona*. We shall adopt the convention that when two countries meet at a single point, we are allowed to color them the same—so *Utah* and *New Mexico* may be colored the same, as may *Colorado* and *Arizona*. This convention is necessary, since otherwise we could construct "pie maps" that require as many colors as we choose—for example, the eight-slice pie map below would need eight colors, eight slices meet at the center. With our convention, this map only two colors.

Another familiar "map" that needs only two colors is the ch At each meeting point of four squares we alternate the colors *white*, producing the usual chessboard coloring (see below).


EULER'S FAMOUS FORMULA 31

If we now relax the condition that the faces must all be of the same type but still require the corners to have the same arrangement of regular polygons around them, then we obtain the *semiregular* (or *Archimedean*) *polyhedra*. There are two infinite families of these, the prisms and the antiprisms, consisting of a pair of congruent polygons on the top and bottom, with a strip of squares or equilateral triangles around the middle.

prism antiprism

There are also thirteen other semiregular polyhedra, some with wonderful names, such as the snub cube and the great rhombicosidodecahedron. Illustrated below are the cuboctahedron (with square and triangular faces), the truncated octahedron (with square and hexagonal faces), the truncated icosahedron (with pentagonal and hexagonal faces), and the great rhombicuboctahedron (with square, hexagonal, and octagonal faces).

cuboctahedron truncated octahedron

truncated icosahedron great rhombicuboctahedron

These polyhedra are not just mathematical curiosities—they are found widely throughout nature: for example, crystals of iron pyrites occur naturally as cubes, octahedra, and dodecahedra, while lead sulphide crystals take the form of cuboctahedra. More recently, certain