Mathematics Model Collection History

The Department of Mathematics began collecting mathematical models in the late 1800s when such model-making was popular, particularly in Germany. More than 380 of these historically and mathematically significant sculptures are currently on display in Altgeld Hall at the University of Illinois at Urbana-Champaign.

In the 1800's, mathematicians exploring the nature of surfaces in space began to construct physical models of those surfaces, as an aid to teaching and research. Some companies, particularly in Germany, began to offer these models commercially. By the end of the century, many American mathematics departments began to develop model collections.

The history of our collection can be traced back to Edgar Townsend, who was hired as an Assistant Professor of Mathematics in 1893. This was the year of the World Columbian Exposition in Chicago, at which the International Congress of Mathematicians was held. Felix Klein traveled to Chicago from Germany and helped showcase the German models then being produced. Six years later, Townsend went to the University of Göttingen, to study under David Hilbert. This was the center of German mathematics and today it has the world’s most complete model collection. Upon completing his dissertation, Townsend returned to Urbana. He served as head of the mathematics department from 1905–1928 during which time he ordered a complete set of German geometric models. In 1911, Arnold Emch joined the faculty and built more models to expand the university's collection.

Edgar J. Townsend with the beginnings of the model collection at the University of Illinois.

As mathematics became more abstract, interest in physical models died out, but the Department of Mathematics at the University of Illinois maintained its strong collection: this is the largest collection of such models on public display, second only to that in Göttingen. In the last decades of the 20th century, computer graphics revived an interest in mathematical visualization. Even more exciting is the process of 3D printing which promises to make it possible to reproduce physical models which were originally made with plaster of Paris.

Arnold Emch

5 lines and 7 lines, 1925 (aluminum and brass)

Details about the two aluminum and brass sculptures in the gallery were first presented in the University of Illinois Bulletin published April 27, 1925. The sculptures were planned and designed in the Mathematical Library of the University of Illinois under the direction of Arnold Emch, Associate Professor of Mathematics. From Professor Emch: “The two models are examples of simple special cases of surfaces by systems of plane algebraic curves determined by the intersections of their planes with certain fixed lines and curves, obtained by assuming three lines *l*, g, h and another fixed line s and the pencil of planes through s. Every plane in the pencil cuts *l*, g, h in three points which determine a circle. The locus of these circles is, in general, a quartic surface.” In this photo the models were on display at CalculArt: Exhibit of Mathematical Art at the Dennos Museum Center at Northwestern Michigan College.

Torricelli's trumpet

A surface of revolution is created by revolving a planar curve around an axis of rotation that lies in the same plane. The famous surface of revolution pictured here is known as Gabriel's horn or Torricelli's trumpet. In cylindrical coordinates (r, θ, z) the surface is defined by the equation $r = 1/z$ and by the inequalities $1 \le z \le H$. Letting H tend to infinity leads to a remarkable situation. The resulting surface has infinite surface area, yet it encloses a finite volume of π cubic units. This possibility is sometimes called the Painter's Paradox, although it is not a paradox at all.

Evangelista Torricelli, a student of Galileo, is perhaps best known for his contributions to physics, such as Torricelli's Law in fluid dynamics and the Torricellian barometer. The name Gabriel's horn refers to the archangel Gabriel.

Peano surface

Guiseppe Peano gave the first rigorous treatment of the theory of maxima and minima of functions of several variables in 1884. His work includes a discussion of what are now known as Peano surfaces. The Peano surface pictured here is defined by an equation of the form

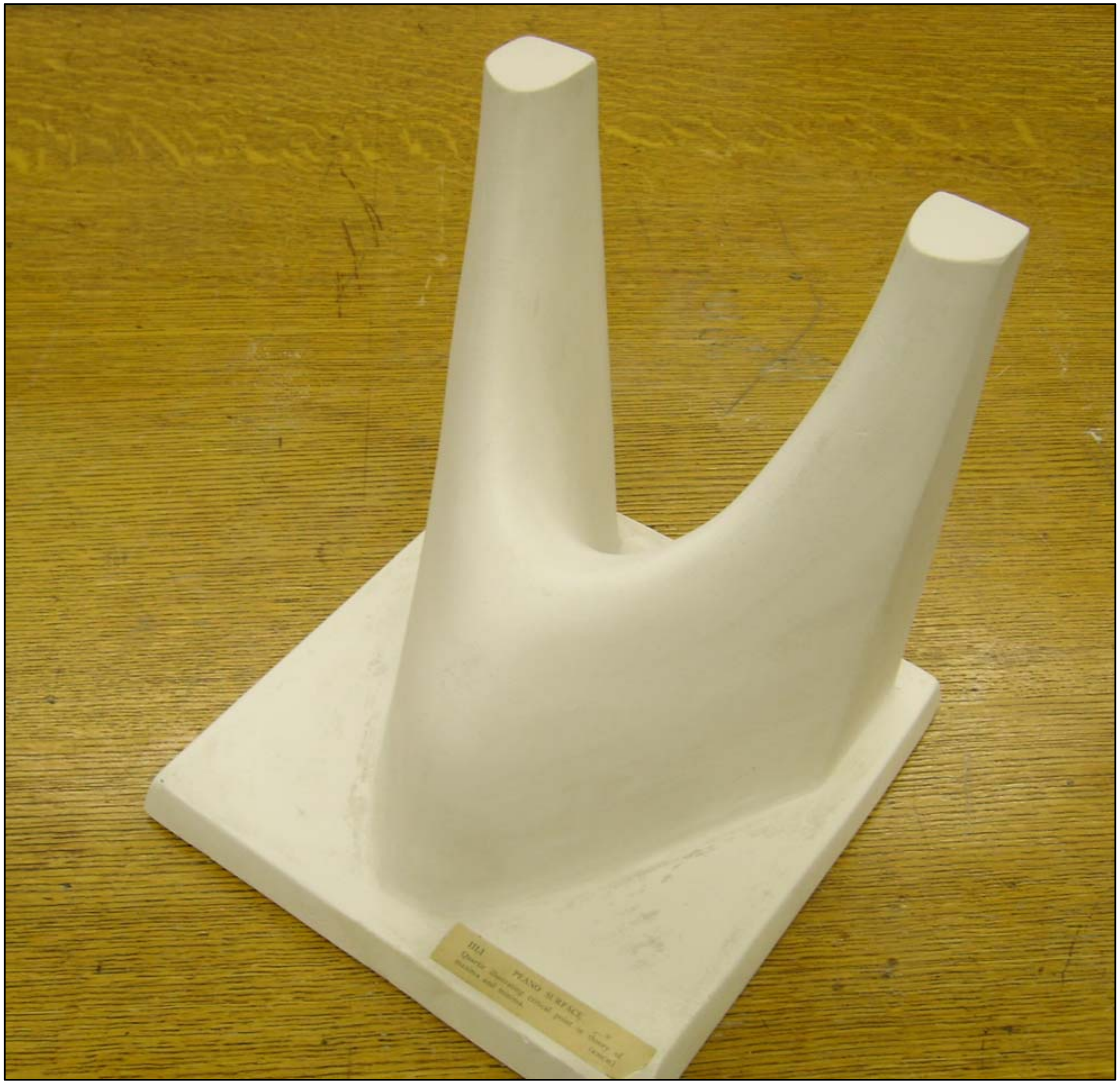
$$z = f(x, y) = (y^2 - ax)(y^2 - bx)$$

where a and b are distinct positive constants. It provided a counterexample to a carelessly stated criterion for a local minimum which was commonly believed in Peano’s time.

The function f satisfies a stunning property. If we restrict f to any line through the origin, then the restriction to that line has a strict minimum at the origin. On the other hand, the function itself **does not have** a minimum at the origin. To see why, first note that the zero set of f consists of the union of two parabolas, touching only at the origin. The function f has negative values in the thin region between these parabolas. Outside of both it has positive values. Hence in every neighborhood of the origin f achieves both signs. A simple geometric argument (or an easy calculation) shows, however, that its restriction to every line through the origin has a strict minimum.

This example is often discussed in Calculus III courses at the University of Illinois. The surface illustrates the situation beautifully.

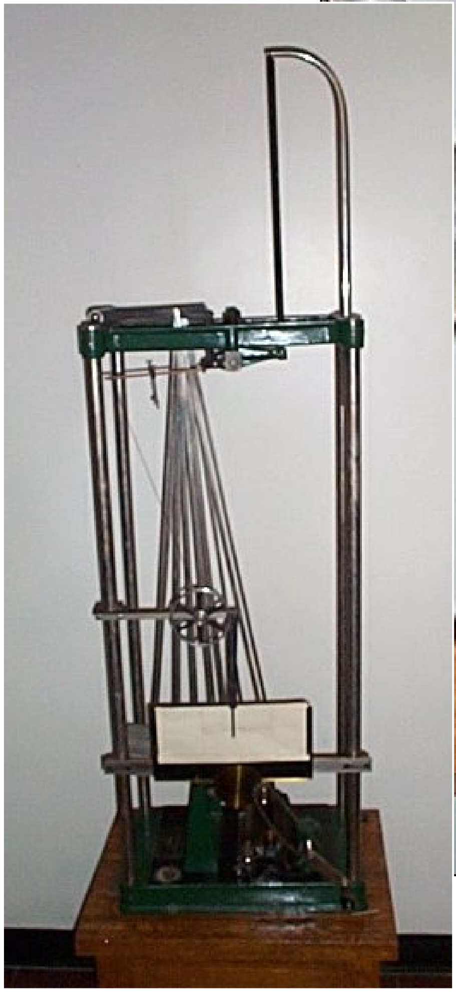
The model shown here was made by Arnold Emch.



Fourier synthesizer

This is a mechanical Fourier synthesizer. The pen position is controlled by 80 springs driven by 80 gear-wheels that turn in relative rates of 1/80, 2/80, 3/80...79/80 and 80/80 turns per turn of the crank. The rods can be moved on the rocker arms to represent the 20 amplitudes of the Fourier coefficients. The 20 rods pull against the 20 springs, which in turn pull against the master spring. The master spring moves the pen and provides the sum of the series.

This machine was constructed by Gaertner and Company of Chicago during the late 19th century. The company also manufactured astronomical and astrophysical measuring instruments for Yerkes Observatory and interferometers for professors at the University of Chicago. After changing ownership several times, the Gaertner Scientific Corporation is now located in Skokie, Illinois, where they continue to design and manufacture precision scientific instruments.



Dandelin's Theorem

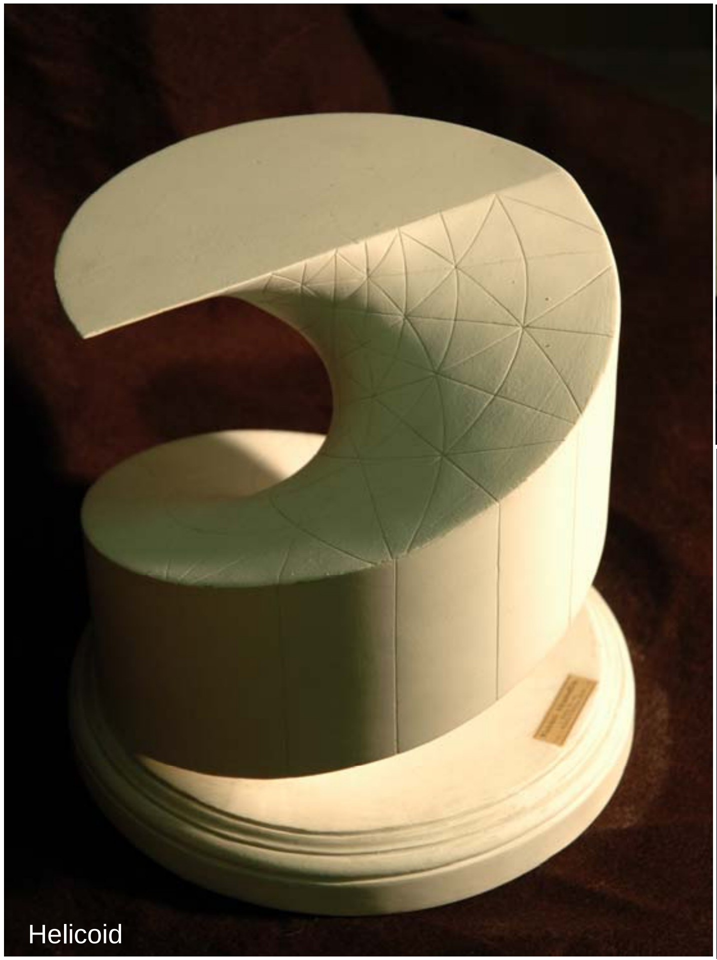
In a footnote in his celebrated 19th century treatise on conic sections, George Salmon proves Germinal Dandelin's theorem, without crediting the Belgian mathematician in this way: From a point outside a sphere, any two segments tangent to the sphere have equal length. Now place two spheres inside a single cone so as to be simultaneously tangent to the same plane cutting the cone in an ellipse. Then from any point on the conic, the spheres will be tangent also to the cone along two horizontal circles which are a fixed distance apart as measured along a generating line of the cone. Pick one such segment on the cone and consider the point where it crosses the ellipse. The distance from this point to the place on the plane where one sphere, say the smaller one, touches the plane, equals the part of the generating segment to the smaller sphere. The other part of the segment is duplicated by the distance from the point on the ellipse to the point where the larger sphere is tangent to the plane. Thus, the sum of the two distances in the plane to where the sphere touches the plane is constant anywhere along the ellipse. Dandelin's sphere touches the plane at the foci of the ellipse. The analogous theorem holds for hyperbolas for a double cone.

This charming wooden model can be manipulated (carefully) by rotating the middle part against the top and bottom held together by the wire generator. The slider moves up and down the wire as it moves around the ellipse; the total length of the string remains constant. The inset shows a computer graphic of a real-time interactive computer animation of this idea which is one of many programmed over the years by future and in-service students of our secondary math education program.

Helicat

The helicoid and the catenoid are among the oldest minimal surfaces, found by Jean Baptiste Meusnier in 1776. A minimal surface has the curvature properties of soap films, locally shaped like a saddle, and shrunk to the smallest area allowed by the global constraints, such as a fixed boundary, or certain behavior at infinity. The helicoid is generated by a straight line perpendicular to an axis, and turning at a constant rate as it moves along the axis. A catenoid is generated by rotating the curve made by a hanging chain, about an axis of rotation. These two harmonic conjugate surfaces are isometric, in the sense that a small patch that fits one snugly will also fit the other. Such a patch is represented for the plaster models by pieces of thin brass, hammered to have the negative curvature of the two surfaces.

What these physical models cannot show is how the helicoid is related to the catenoid by a one parameter family of isometric surfaces. The bottom image shows Do Carmo's parametrization of this deformation which fits directly into *Mathematica*. You can program this formula into other common graphing utilities. The first three entries are the xyz-coordinates and the last three the ranges of the uvt-parameters. As it progresses the helicoid contracts to finally wrap around the catenoid infinitely often.


Helicoid


Brass patch


Catenoid

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In[21]:= Manipulate[
  ParametricPlot3D[{(Cos[t] Cosh[v] + Sin[t] Sinh[v]) Cos[u],
    (Cos[t] Cosh[v] + Sin[t] Sinh[v]) Sin[u], v Cos[t] + u Sin[t]},
   {u, 0, 2 π}, {v, -1, 1}], {t, 0, 2 π}]
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Orthanc

This month's surface has been inventively dubbed “Orthanc” by Abby Watt, for lack of any idea what Arnold Emch meant by its label: “Cauchy surface $z' = z^3 - 1$". Abby is the University of Illinois art student whose project it was to capture selected plaster models [October-Boy, November-Kuen, August-Cauchy] with a *Next Engine 3D Scanner* in preparation for duplicating the models with a *ZCorporation ZPrinter 406*. The smaller of the twins shown is the original plaster model “designed and constructed in the mathematical department of the University of Illinois", as Professor Emch puts it in a *University of Illinois Bulletin* of November 11, 1920. Its larger sibling is a nearly perfect reproduction, down to the (barely) readable label.

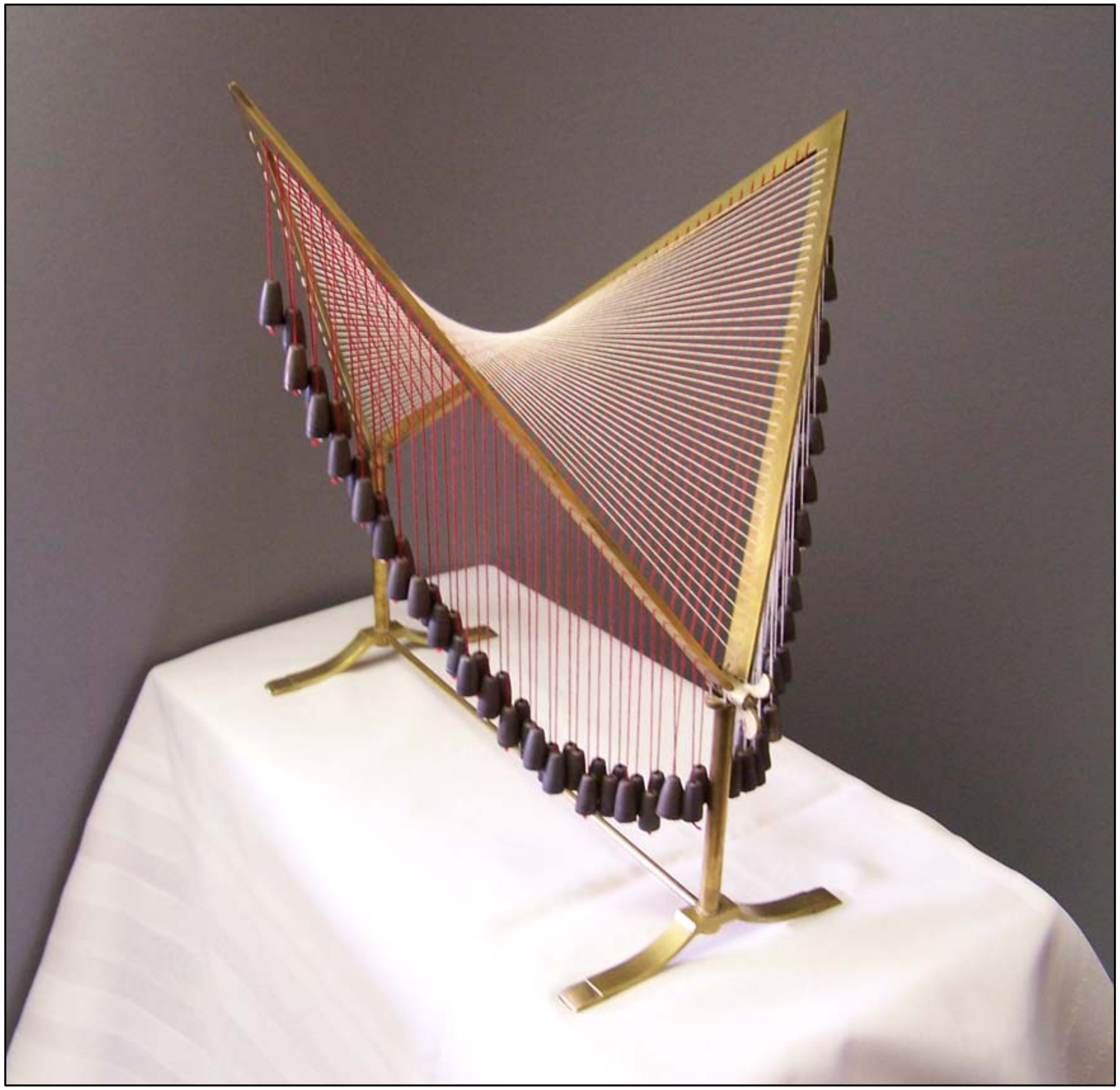
A Cauchy surface, according to Emch, is an imaginative graphing of a complex analytical function, here $f(z) = z^3 - 1$, by mapping $|f(z)|^2$ to the third dimension above the z-plane. In Cartesian xyz-coordinates Orthanc has equation

$$z = -(x^2 + y^2)^3 + 2x^3 - 6xy^2 - 1$$

worthy of a final examination problem in Calculus III. The significance of visualizing complex function graphs in this manner is now lost in the midst of the nineteenth century. Today we can navigate 4D on our laptops to explore complex function graphs directly.

Hyperbolic Paraboloid

Professor Emeritus Robert Fossum recently restored this string model which gives a doubly ruled surface, in this case a hyperbolic paraboloid. The model consists of two "wings" each of which can move about the fixed axis formed by the tops of the two base feet. The threads in the model had deteriorated over time and many had broken. The base and frame are made of copper or bronze which had become blackened. After cleaning the metal, Fossum used cotton perle thread to restring the model. The holes are quite small, so a very thin needle had to be used. The weights are common fishing line sinkers (just as was used in the original model), but small squeezable sinkers were used to hold the larger sinkers in place. It took Fossum several hours to re-string this model.



A ruled surface is obtained by connecting "corresponding points" on two curves by a line. Suppose that two curves C(s), D(s) in space parameterized by s are given (and not coplanar). For each s, let L(s) be the line determined by the two points (C(s), D(s)). Then the surface obtained is called a ruled surface. One simple parameterization is given by

$$R(s,t) = C(s) + (D(s) - C(s))t$$

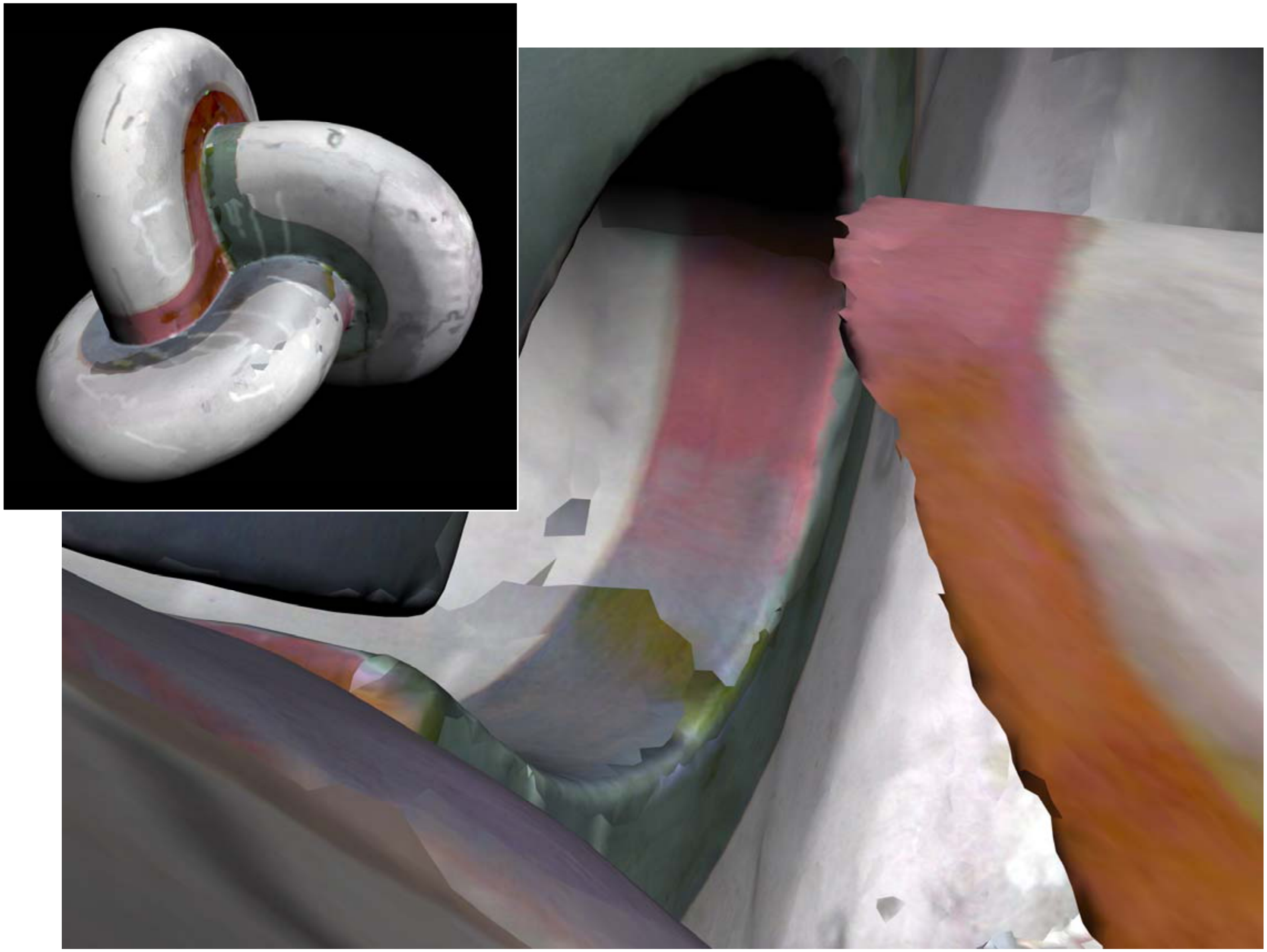
In the case of the hyperbolic paraboloid, the two curves are both straight lines which are the opposite sides of the corresponding rectangle when the wings are in the plane. Two rulings are made by connecting each pair of opposite sides.

As the angle is reduced from 180° to 0° by rotating the wings toward each other, one gets a family of hyperbolic paraboloids, with the limiting cases of a plane and a double-covered parabola.

Boy Surface

Werner Boy produced two immersions of the projective plane in 3-space in connection with his doctoral dissertation directed by David Hilbert, circa 1901. Topologically, the real projective plane is realized by sewing the boundary of a Möbius band to the boundary of a disc. Like the Klein bottle, obtained by sewing two Möbius bands together along their boundary, the projective plane is non-orientable. Thus it cannot be placed into 3-space without self-intersections (if it were, we could fill its inside with paint, and so show that it is 2-sided, like a sphere).

Singular realizations in 3-space of both surfaces were well known by the end of the 19th century. Hilbert thought that unlike for Klein's surface, such an immersion of the projective plane was impossible. Boy's 2 solutions are deformable into each other and have been a popular subject for models, illustrations, analytic parameterizations and computer animations.



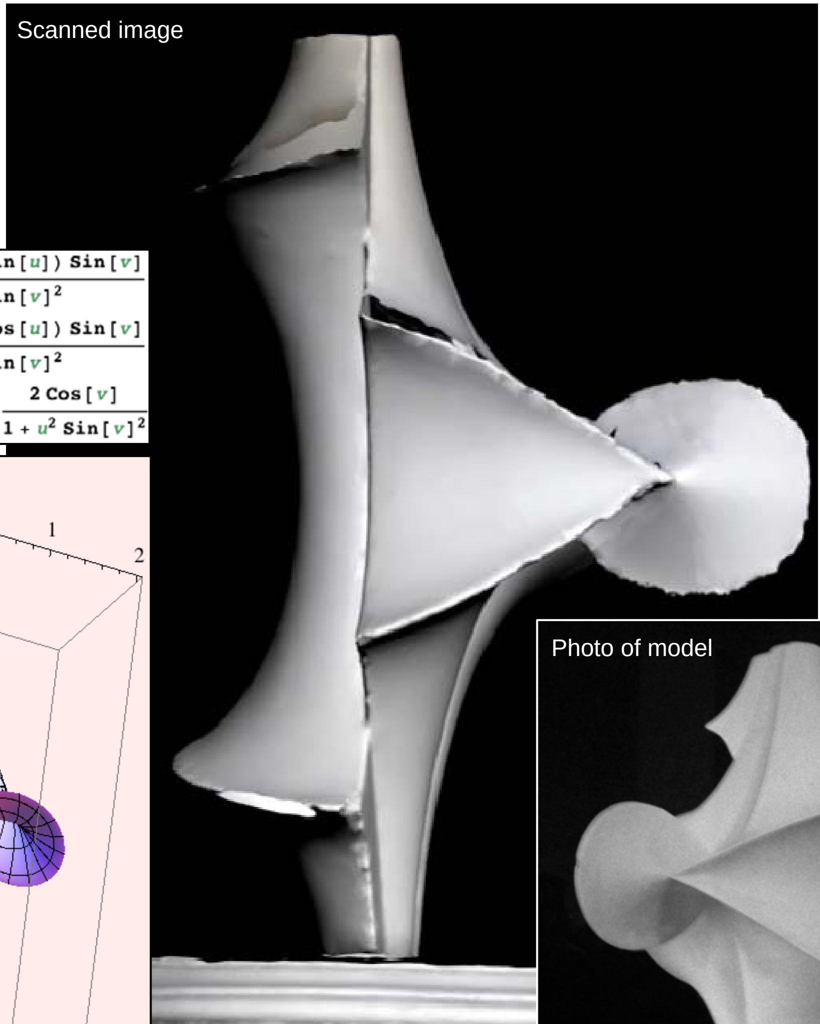
What you see here is not exactly a photograph of the century old model in the Altgeld collection. Rather, it is an evidently imperfect computer reconstruction of a scan. A perfect scan could be used to program a 3D-printer, and thus to reproduce an exact replica of the original. The reconstruction is a surface and so we can use an immersive virtual environment, like the CAVE, to fly about the inside of this very solid plaster object!

Kuen's surface

The most beautiful model in the Altgeld collection is by all accounts Kuen's surface. It is located right next to the main office in our model cases. It is a surface of constant negative Gaussian curvature with razor sharp edges and complicated self-intersections. It is related to the humbler pseudo-sphere [see March 2009] by a Bianchi transform. The deep mysteries of a corner of conformal geometry captured by this model continue to interest contemporary mathematicians as well as computer graphicists who find its delicate structure a challenge to render.

As you can see, one of the images is a photograph of the model, one is a digital reproduction using a scanner, and the color image renders Alfred Gray's formulas using Ulises Cervantes-Pimentel's adaptive mesh smoothing algorithms embedded in *Mathematica* 6.0. To create a 3D print reproducing a plaster model requires an exact description of each horizontal layer of the surface. That is how contemporary 3D-printers work: they follow the same layer-cake strategy as the 19th century modelers did in plaster. The modelers graphed the formulas describing the surface and cut out cross-sections and profiles as cardboard splines. Approximate layers of plaster were stacked on top of each other and then smoothed by hand, using the splines as guides.

Today, descriptions of the layers can be arrived at by explicit formulas, or less tediously, by scanning the plaster original. The scans, however, are rough like the historical plaster models; they must be smoothed. The scan shown here was done by Abby Watt on a $2500 scanner. There is still much experimentation ahead before data adequate for a 3D print of Kuen's surface is available.



Scanned image

$$x[u\_, v\_] := \frac{2\,(\mathrm{Cos}[u] + u\,\mathrm{Sin}[u])\,\mathrm{Sin}[v]}{1 + u^2\,\mathrm{Sin}[v]^2}$$

$$y[u\_, v\_] := \frac{2\,(\mathrm{Sin}[u] - u\,\mathrm{Cos}[u])\,\mathrm{Sin}[v]}{1 + u^2\,\mathrm{Sin}[v]^2}$$

$$z[u\_, v\_] := \mathrm{Log}\left[\mathrm{Tan}\left[\frac{v}{2}\right]\right] + \frac{2\,\mathrm{Cos}[v]}{1 + u^2\,\mathrm{Sin}[v]^2}$$

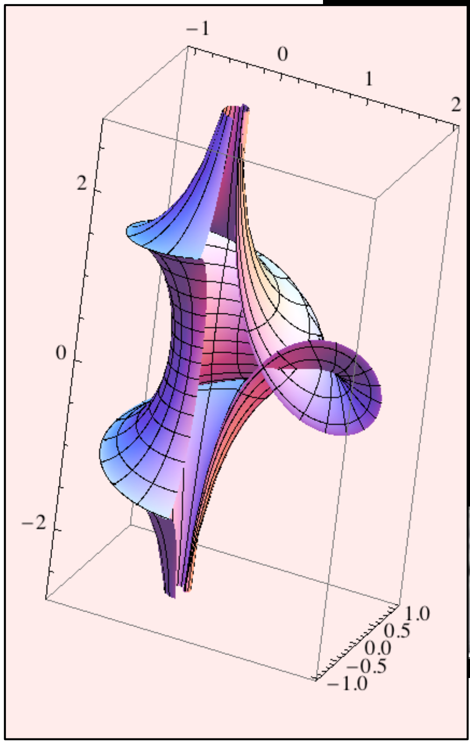


Photo of model

Future of the math model collection

We hope you have enjoyed the 2009 calendar and that our enthusiasm for the models has proved infectious. We are fortunate that the models, bought from abroad or made at Illinois, were carefully preserved in their glass cases for nearly a century. However, preservation is not enough because they deteriorate over time, as you can see from the two string models shown here.

So preservation must include restoration. This is no trivial task. The Smithsonian Institute in Washington D.C. has a vast collection of mathematical models, but almost all of them are archived in the basement. Two string models that were restrung some years ago and put on display in the museum cost $45,000 to restore. Obviously, we hope to accomplish this task in a more economical manner. The goal over the coming year is to figure out just how to accomplish the restringing. Thereafter, we expect to reach a point where undergraduate math students can do this as they learn the history and mathematics residing in these unique objects.

The plaster models deteriorate more slowly. Many can be restored to their original luster by gently rubbing them with cheese cloth. Unfortunately, less benign restoration attempts in the distant past damaged some of them beyond the scope of this simple remedy. These we hope to recreate using 3D printing techniques. Once we have figured out how to generate the numerical data for the printer, a full size replica can be made for a few hundred dollars worth of material. At the same time, however, we can print inch-high copies in bulk which can be used for promotional purposes.

Some time in the future we may be able to produce another calendar showing the fruits of our restoration efforts.

Listed here are the many people who helped make this calendar possible:

Designer and Editor: Tori Corkery

Picture credits:

- Kalev Leetaru: March, July
- George Francis: February, April, June, December
- Tori Corkery: September, November
- Abby Watt, University of Illinois student: October, November
- John Sullivan, Professor at the Technical University of Berlin: May

Assisting the editor with writing, proofing, and many factual details were: John D'Angelo, Robert Fossum, George Francis, Bruce Reznick, Milos Curcic, Jiri Lebl, Wendy Harris, Jared Bronski, and Sara Nelson, all from the Department of Mathematics at the University of Illinois; and Hank Kaczmarski of Integrated Systems Lab of the Beckman Institute, University of Illinois.