Math 595 MNA Spring 2019
Methods in Nonlinear Analysis and Applications to Differential Equations

Class time: TR 2 – 3:20pm
Lecturer: Eduard Kirr, e-mail: ekirr@illinois.edu
Description: The first part of the course will focus on fixed point theorems based on *degree theory* and their applications to differential equations. The (Brouwer) degree for maps between finite dimensional spaces has deep roots in homotopy and homology and you might have encountered it when studying these theories. While I will briefly discuss how it is introduced there, I will present a more axiomatic view of degree theory which can be generalized to (nonlinear) maps between infinite Banach spaces having certain compactness properties (completely continuous, properly bounded, etc.). I will discuss its powerful consequences: *Brouwer, Borsuk-Ulam, Ham and Sandwich Theorems* in finite dimensions and the *Leray-Schauder type fixed point theorems* in infinite dimensions. Their applications to modern *local and global bifurcation theories* and to solutions of differential equations will be emphasized. This part of the course will culminate with the contribution made by the degree theory in understanding the *collapse of Tacoma-Narrows bridge,* a phenomenon that was not predicted by laboratory simulations or the structural and dynamical stability theory preceding its construction.

The second part of the course will focus on *contraction principle* and *variational methods.* While independent, this part will extend some of the results in the first part to non-completely continuous (non-compact) maps. Metric and Banach spaces will be reviewed and *Calculus in Banach spaces* will be introduced before proving the Banach fixed point theorem and its consequences e.g., the implicit function theorem (IFT) in (infinite) dimensional Banach spaces. The application of the contraction principle to existence, uniqueness, continuous dependence of data and stability for solutions of evolution equations (including systems of ordinary and partial differential equations) will be briefly discussed while the applications of the IFT to *Lyapunov-Schmidt decomposition* and *local bifurcation theory* will be presented in detail with examples from nonlinear optics and statistical physics. Extensions of the local results to non-perturbative regimes via the *global bifurcation theory* for real analytical maps will also be discussed. The variational methods will be based on the rigorous calculus in Banach spaces which replaces the more common but ad-hoc "calculus of variation" in which the definition of "variation" seems to change from problem to problem. Moreover, when it comes to applications in finding certain equilibria or periodic solutions in partial different equations, the variational methods have to cope with non-convex functional and non-compact constraints. We will discuss how to compensate for these shortcomings via *Rellich or concentration compactness* and then apply the classical theory which of course will be introduced.
References: I will follow my own notes (posted online) based on the following references:

1. Topics in Nonlinear Functional Analysis by L. Nirenberg.

2. Leray-Schauder degree: a half century of extensions and applications by J. Mawhin in Topol. Methods Nonlinear Anal. Volume 14, Number 2 (1999), 195-228.

3. Partial Differential Equation by Lawrence C. Evans.

4. Long time dynamics and coherent states in nonlinear wave equations by E. Kirr in Recent Progress and Modern Challenges in Applied Mathematics, Modeling and Computational Science, R. Melnik, R. Makarov, J. Belair, Eds. in Fields Institute Communications, 2017, pp. 59–88.

5. Large Torsional Oscillations in a Suspension Bridge: Multiple Periodic Solutions to a Nonlinear Wave Equation by K.S. Moore in SIAM J. Math. Anal. Vol. 33, No. 6, pp. 1411-1429

Grading Policy: There are no homework assignments or exams for this course. The participants will be asked to make a presentation on the applications of these techniques to a nonlinear differential equation, preferably from their own research area. Grades will be based on class activity, and on the quality of the presentation.