MATH 595: CATEGORICAL LOGIC (FALL 2019)

Instructor: Ruiyuan (Ronnie) Chen

Time and location: TBD

Course description: Broadly speaking, a logic consists of syntactical expressions (e.g., "\( \forall x \neq 0 \exists y (xy = 1) \)"), rules for manipulating them, and rules for giving them mathematical meaning. Categorical logic provides a framework for addressing the question: how well does the syntax of a logic match its semantics? That is, is every semantic "thing" (be it truth or data) syntactically expressible, and irredundantly so? In the best cases, the syntax of a logic is encoded in an algebraic structure\(^1\) which is precisely equivalent to the space of all possible semantics via a duality theorem. We will study several instances of this phenomenon, including

- Stone duality for propositional logic/Boolean algebras/Stone spaces;
- Gabriel–Ulmer duality for Cartesian logic/finitely complete categories/locally finitely presentable categories;
- Makkai duality for first-order logic/pretoposes/ultracategories;
- (time permitting) Joyal–Tierney representation theory, \( L_{\text{ku}} \), and Grothendieck toposes, and/or other topics of interest to participants.

Each such duality theorem manifests simultaneously as a very strong completeness/definability\(^2\) theorem for the logic, a representation theorem for the syntactic algebra, and an axiomatization of the "function algebra" on the corresponding space of possible semantics.

Prerequisites:
- being comfortable with general abstract structures (monoids, posets, etc.) and related notions (homomorphisms, presentations, etc.)
- basic familiarity with point-set topology
- some background in category theory would be helpful but will not be assumed; we will cover everything needed during the course
- ditto for model theory

Coursework: There will be occasional (optional, but highly recommended) exercises. Grading will be based on attendance.

References: I will be posting course notes online. Other good references:
- P. Johnstone, Sketches of an elephant, Part D.
- B. Jacobs, Categorical logic and type theory.
- J. Lurie, lecture notes.
- A. Pillay and friends, lecture notes.

\(^1\)versions include the "Lindenbaum–Tarski algebra" and "syntactic category"

\(^2\)sometimes called “strong conceptual completeness”