Extinction of Family Name

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Description

- The purpose of this project is to give a brief introduction of the Galton-Watson-Bienaymé branching process and develop a numerical example for calculating the extinction probability of American male lines of decent.

- Learning Objectives:
  - Learn about Galton-Watson Binaymé branching process
  - Develop examples of probability generating function

Introduction

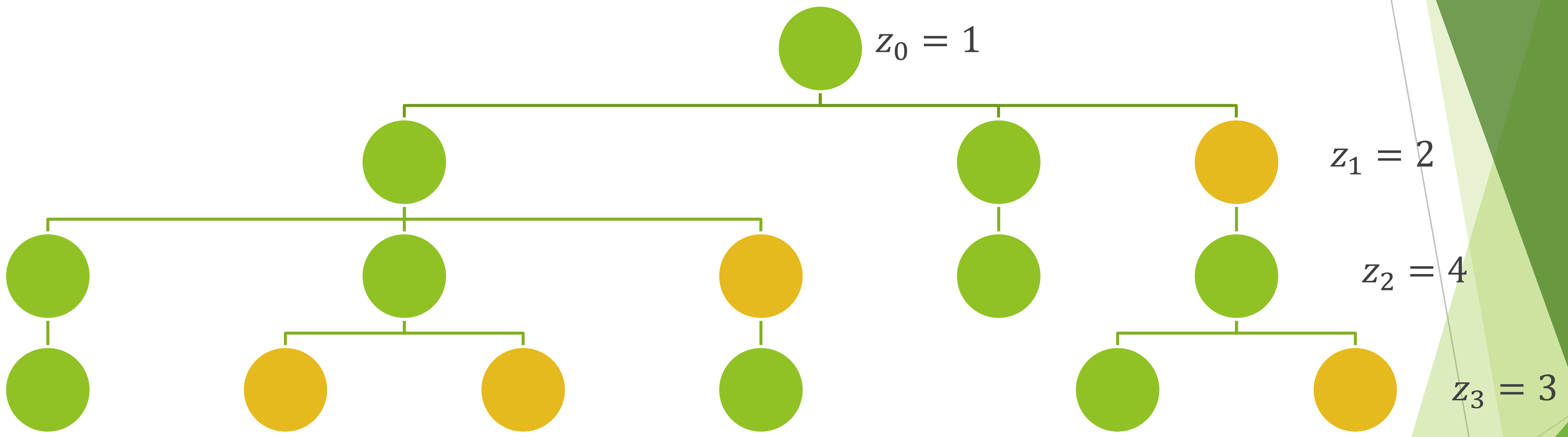
- Suppose that one man starts a new family name. Let

$$z_n = \text{the number of sons in the n-th generation, n} = 0, 1, 2, \ldots$$

- Then $Z_0 = 1$. Lex X be a generic random variable standing for the number of sons from a male in the family and p_k be the probability that a male has k sons, i.e. $P(x = k) = p_k$ for every k = 0, 1, 2,…

- The probability generating function(pgf) P(s) is defined by

$$P(s) = P_X(s) = E(s^x) = \sum_{k=0}^{\infty} p_k s^k$$



This family tree demonstrates the example, with green circles represent the sons and orange circles represent the daughters.

- We shall write
  - $P_n(s) = the\ pgf\ of\ Z_n$, n = 0, 1, 2,…
  - m = E(X) = P'(1) = the number of sons from a given male
  - σ^2 = V(X) = P''(1) + P'(1) – $(P'(1))^2$ = the variance of the number of sons from a given male

The stochastic process {Z_n,n≥0} is called the Galton-Watson-Bienaymé branching process.

- Here are some useful properties of the pgf.
  - If X and Y are independent, $P_{X+Y}(S) = P_X(S)P_Y(S)$
  - If $X_1, X_2, X_3, \ldots, X_n$ are independent random variables, and if $S_n = X_1 + X_2 + X_3 + \ldots + X_n$

$$P_{S_n}(s) = P_{S_1}(s)\ P_{S_2}(s) \ldots P_{S_n}(s)$$

  - Let $X_1, X_2, X_3, \ldots, X_n$ be i.i.d. random variables, and N be a random variable independent of the $X_i's$. Let the random variable $S_N = \sum_{k=1}^{N} X_k$, then

$$P_{S_N}(s) = P_N\left(P_X(s)\right)$$

- We carefully examine how the n-th generation carry forward the family name to the n+1 generation. Let us label the males in the n-th generation by 1,2,3,…,Z_n and let X_i be the number of sons from the male with the label i. Then the total number of males in the n+1 generation would be

$$Z_{n+1} = X_1 + X_2 + X_3 + \ldots + X_{Z_n}$$

- By assumption, the $X_i's$ are independent of each other and of Z_n. Hence by definition, we have the fundamental equation

$$P_{n+1}(s) = P_n\big(P(s)\big) = P\big(P_n(s)\big), \qquad n = 0, 1, 2, \ldots$$

- Suppose that the probabilities $p_k's$ are given by

$$p_k = pq^k, \qquad k = 0, 1, 2, \ldots$$

Then we also have

$$P(s) = \sum_{k=0}^{\infty} pq^k s^k = \frac{p}{1-qs}$$

$$P_2(s) = P\big(P_1(s)\big) = \frac{p}{1 - \frac{qp}{1-qs}} = \frac{p(1-qs)}{1-qs-pq}$$

- We can also use the fundamental equation to calculate moments of Z_n for any n.

If m $< \infty$, then $E(Z_n) = m^n$ for all $n \geq 0$

If $\sigma^2 < \infty$, then $V(Z_n) = \begin{cases} \frac{\sigma^2 m^n (m^n - 1)}{m^2 - m}, if\ m \neq 1 \\ n\sigma^2, if\ m = 1 \end{cases}$

Probability Distribution of the Number of Descendants

- In this section, we apply the methodology developed in the previous section to actual data and calculate the probability distribution of descendants in a female line. We will show by example of how to compute the iterative function from the previous section into matrix form.

- Define c_j to be the proportion of women who have j = 0, 1, … children. Let g define the proportion of births which are girls. Then, of the c_1 mothers with one child, the proportion of daughters is gc_1. Of the c_2 mothers with two children, we have g^2c_2 have two daughters, 2g(1-g) c_2 with one daughter and $(1-g)^2c_2$ with no daughters. Table 1 shows the proportions that are divided into classes.

		Number of daughter			
Total Children	No. of Women	0	1	2	…
0	c_0	c_0	0	0	…
1	c_1	$(1-g)\,c_1$	c_1g	0	…
2	c_2	$(1-g)^2c_2$	$2g(1-g)c_2$	g^2c_2	…
3	c_3	$(1-g)^3c_3$	$3g(1-g)^2c_3$	$3g^2(1-g)c_3$	…
4	c_4	$(1-g)^4c_4$	$4g(1-g)^3c_4$	$6g^2(1-g)^2c_4$	…
…	…	…	…	…	…

Table 1. Distribution of women according to total children and number of daughters.

- Table 2 summarizes the total female population in China by number of children. A frequently used measure in demographic studies is the sex ratio at birth(SRB) which is the number of boy infants compared to girl infants who are born within a given period usually represented by the number of boys per 100 girl infant. According to the Chinese 2000 Population Census, the SRB is 119.9.
- Table 3 shows the distribution of women according to total number of children and number of daughters using Table 2 and the SRB.

	Number of Children					
	0	1	2	3	4	5
No. of Women	9,080,779	11,519,885	8,548,763	3,187,520	895,589	315,130

Table 2. China Female Population by Number of Children
*Source: United Nations Demographic Yearbook 2000

	Number of Daughter						
Total Children	c_i	0	1	2	3	4	5
0	0.27068	0.27068	0.00000	0.00000	0.00000	0.00000	0.00000
1	0.34339	0.18723	0.15616	0.00000	0.00000	0.00000	0.00000
2	0.25482	0.07576	0.12637	0.05270	0.00000	0.00000	0.00000
3	0.09501	0.01540	0.03854	0.03214	0.00894	0.00000	0.00000
4	0.02670	0.00236	0.00787	0.00985	0.00548	0.00114	0.00000
5	0.00939	0.00045	0.00189	0.00315	0.00263	0.00110	0.00018

Table 3. Distribution of women according to total number of children and number of daughters.

*According to the SRB, g = 100/(100+119.9) = 0.45475

*$c_i = \frac{\text{number of women with i children}}{\text{total number of women}}$

Probability of Eventual Extinction

- $P0 = 0$

Every generation has at least one descendant → probability of extinction = 0

  - $P1 = 1$: one descent every generation
  - $P1 < 1$: $Zn \rightarrow$ infinite

Probability of Eventual distinction

- $P0 > 0$
  - $\{Zn=0\}$ is a subset of $\{Zn+1 = 0\}$
  - $\{extinction\} = \{ Zn = 0 \text{ for some } n >=1\} =$
  - Continuity property

    $\xi = P\ (extinction) = P\ (\lim_{n \to \infty} U \{ Z_k = 0 \}) = P\ (\lim_{n \to \infty} \{ Z_n = 0\})$

    $= \lim_{n \to \infty} P\{ Z_n = 0\})$ ..

    * P(Z_n =0 ） increase as n increases

Probability of Eventual distinction

- P0>0
  - $X_n = P_n(0) = P(Z_n = 0)$ for $n \geq 0$
  - $X_{n+1} = P_{n+1}(0) = P(P_n(0) = p(x_n))$
  - Combine the continuity property and the property of X_{n+1}
    - $\xi = \lim_{n \to \infty} X_{n+1} = \lim_{n \to \infty} P(X_n) = P(\xi)$
    - $\rightarrow \xi$ has to be the smallest root of $P(S) = S$