COMBINATORICS COMPREHENSIVE - Spring 2007

Submit *only* FIVE problems from Part I and THREE problems from Part II; 80 points possible. Passing requires good performance on each Part. Justify answers; GIVE CLEAR STATEMENTS of any theorems you use.

Part I

1. During $2n$ flips of a fair coin, a running total of heads and tails is kept. Compute the probability that the lead changes (one type leads and later the other type leads), given that each outcome appears n times.

2. Let a_n be the number of n-tuples in $[4]^n$ that have at least one 1 and have no 2 appearing before the first 1 (note that $\langle a \rangle$ begins $0, 1, 6, \ldots$). Obtain and solve a recurrence for $\langle a \rangle$. Give a direct counting argument (without using summations) to prove the resulting simple formula.

3. Let $\langle a \rangle$ be the sequence whose generating function is $\frac{3-3x}{1-x-2x^2}$. Without obtaining a formula for a_k, obtain from the generating function a simple formula for $\sum_{k=0}^{n} a_k$ as a function of n.

4. Derive a summation formula whose value is the number of permutations of $[n]$ with no cycles of length 2. *Explain* why the formula is correct.

5. A *tournament* is an orientation of a complete graph, and the *outdegree* of a vertex in a tournament is the number of edges leaving it. Apply Hall's Theorem to prove that there is a tournament with outdegrees $d_1, \ldots, d_n$ if and only if for each $k \in [n]$ the k smallest of these numbers sum to at least $\binom{k}{2}$, with equality when $k = n$. (Hint: Construct a graph H with $2\binom{n}{2}$ vertices such that a perfect matching in H yields the desired tournament.)

6. The *Kneser graph* $K(n,k)$ has vertex set $\binom{[n]}{k}$, with two vertices adjacent if they are disjoint k-sets. Prove that $\chi(K(n,k)) \le n - 2k + 2$ by covering the vertices with $n - 2k + 2$ independent sets. Prove that this is optimal when $n = 2k + 1$. (Example: The Petersen graph is $K(5,2)$. Comment: In fact equality always holds.)

7. Use Euler's Formula to count the regions determined by a configuration of n lines in the plane, where no three lines have a common point.

Part II

8. Let $b_1, \ldots, b_n$ be distinct integers. Show that some nonempty subset of $\{b_1, \ldots, b_n\}$ has sum divisible by n. Determine whether the same conclusion holds for every set of $n - 1$ distinct integers.

9. Use partially ordered sets to prove that every list of $mn+1$ distinct integers has an increasing sublist with more than m elements or a decreasing sublist with more than n elements. (Note: A sublist of a list need not occupy consecutive positions.)

10. Consider the random graph G generated with edge probability p, where p is a function of n. Prove that if $pn \to 0$, then almost always G has no cycles. Use a *different* random variable to prove that if $pn \to \infty$, then almost always G has a cycle

11. Explain how to construct a pair of orthogonal latin squares of order 15. Include all needed building blocks, but do not write out the final pair of squares.