COMBINATORICS COMPREHENSIVE - Spring 2006

Do FIVE problems from Part I and THREE problems from Part II. Passing requires adequate performance on each Part. Answers must be justified in words; give clear statements of any theorems you use.

Part I

1. Form a list $b_1, \ldots, b_n$ as follows: for each k, choose b_k from $\{1, \ldots, k\}$, with each value being equally likely. Obtain a simple formula for the probability that the list is nondecreasing. (Hint: Transform the problem to permit the use of a well-known formula.)

2. Let $\langle a \rangle$ satisfy $a_n = 3a_{n-1} - 2a_{n-2} + 2^n$ for $n \geq 2$, with $a_0 = a_1 = 1$. Express the generating function for $\langle a \rangle$ as a ratio of two polynomials. Obtain a formula for a_n as a function of n.

3. Suppose we roll a six-sided die until each of the numbers one through five have appeared at least once. What is the probability that we succeed sometime during the first n rolls?

4. The squares of a 4-by-4 chessboard are being painted (on the front only), with k colors being available for each square, but the four corner squares must not all be given the same color. Determine the number of distinguishable ways to paint the squares.

5. Let S be a set of permutations of $[n]$. Prove that if $|S| \leq n/2$, then some permutation of $[n]$ differs in every position from every member of S. (Hint: Model this using a graph problem.)

6. The *rhombicosadodecahedron* is a polyhedron in which every vertex is incident to one triangular face, one pentagonal face, and two (opposite) quadrilateral faces. Determine the number of faces in the rhombicosadodecahedron.

Part II

7. Prove that the graph Ramsey number $R(mK_2, mK_2)$ equals $3m - 1$.

8. Use a result on partially ordered sets to prove that every list of $mn+1$ distinct integers has an increasing sublist with $m+1$ elements or a decreasing sublist with $n+1$ elements.

9. Let H be a hypergraph with m edges in which every edge has at least k vertices. Recall that if $m < 2^{k-1}$, then H is 2-colorable. Regardless of the value of m, prove that if each edge intersects fewer than 2^{k-3} other edges, then H is 2-colorable.

10. Consider the random graph model $\mathbb{G}(n, p)$, where $p = o(1/n)$. Prove that in this model the random graph almost always has no cycles.