U. OF ILLINOIS COMBINATORICS COMPREHENSIVE EXAM – SPRING 2018

Submit exactly THREE problems from each Part. Passing requires good performance on each Part. Give full explanations, including CLEAR STATEMENTS of any theorems that you use. Formulas without explanations do not receive much credit.

No external assistance permitted.

PART I

1. Let A_n be the number of permutations σ of $1, 2, \ldots, n$ such that $\pi(i) \neq i + 1$ for $1 \leq i \leq n - 1$. For example, for $n = 3$, the three permutations (in two-line notation) satisfying this condition are

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

Separately, let B_n be the number of permutations π on $1, 2, \ldots, n$ such that in one-line notation, there is no consecutive "$i\ i + 1$" for $1 \leq i \leq n - 1$. For example, for $n = 3$, the three permutations satisfying this condition are

$$132, \ 213, \ 321.$$

Using inclusion-exclusion or otherwise, prove $A_n = B_n$ for $n \geq 1$.

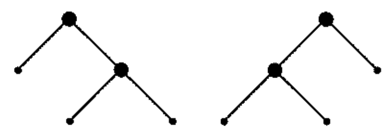
2. Let P_n be the number of integer partitions of n where the parts differ by at least 2. For example, if $n = 22$ then $\lambda = (9, 7, 4, 2)$ counts but $\lambda = (9, 7, 6)$ does not. By considering the Ferrers' shapes for such partitions (or otherwise) prove the following

$$\sum_{n \geq 0} P_n x^n = \sum_{k=0}^{\infty} \frac{x^{k^2}}{(1 - x)(1 - x^2) \ldots (1 - x^k)}.$$

3. Let m, n be two positive integers. A function $f : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$ is *very surjective* if $|f^{-1}(j)| \geq 2$ for $1 \leq j \leq n$. Using exponential generating series (or otherwise) determine an explicit summation formula for the number of very surjective functions.

4. A *full binary tree* is a rooted tree where every vertex other than a leaf has exactly two children (a left child and a right child). Thus, if such a tree has n non-leaves, it will have $n + 1$ leaves. Let T_n be the number of such trees with n non-leaves (equivalently $n + 1$ leaves). Prove that $T_n = C_n = \frac{1}{n+1}\binom{2n}{n}$ (the Catalan number). (Continued next page.)

For example, when $n = 2$ there are two such trees, as depicted below. (The non-leaves are marked with a thicker dot.) This agrees with $C_2 = \frac{1}{3}\binom{4}{2} = 2$:



PART II

5. Let G be an n-vertex simple, connected planar graph, with shortest cycle of length at most 10. At most how many edges of G can have?

6. State and prove Hall's Theorem.

7. Let G be a connected 5-regular graph with exactly one cut edge xy. Let G_1 and G_2 be the components of $G - xy$. Prove that if G_1 and G_2 are 4-edge-connected, then G has a perfect matching.

8. State and prove Mantel's Theorem. [Determine the maximum number edges of an n-vertex triangle-free simple graph.]

PART III

9. Prove that every set of n integers has a subset summing to a multiple of n.

10. Prove or disprove:
(a) Every graph G with chromatic number k has a proper k-coloring in which some color class has $\alpha(G)$ vertices, where $\alpha(G)$ denotes the independence number of G.
(b) If G is a connected graph, then the chromatic number of G is at most $1 + a(G)$, where $a(G)$ is the average of the vertex degrees of G. In other words, $a(G) = 2|E(G)|/|V(G)|$.

11. Prove that for large n with probability at least 1/2, the maximum length of a constant consecutive string in a random list of n heads and tails is at least $0.9 \ln n$ and at most $1.1 \ln n$.

12. Explain how to construct a pair of orthogonal Latin squares of order 15. Include all needed building blocks but do not write out the final pair of matrices.