

U. OF ILLINOIS COMBINATORICS COMPREHENSIVE
EXAM – SPRING 2012

Submit exactly THREE problems from each Part. Passing requires good performance on each Part. Give full explanations, including CLEAR STATEMENTS of any theorems that you use. Formulas without explanations do not receive much credit.

No external assistance permitted.

PART I

1. Use generating series methods to prove the identity below

$$\sum_{k \geq 0} \binom{n}{k} \binom{2k}{k} = \sum_{k \geq 0} \binom{n}{2k} \binom{2k}{k} 3^{n-2k}$$

Hint: What is the generating series for $\sum_{k \geq 0} \binom{2k}{k} y^k$?

2. During $2n$ flips of a fair coin, a running total of heads and tails is kept. Compute the probability that the lead changes (one type leads and later the other type leads), given that each outcome appears n times.

3. Let t_n be the number of ways that n children can be arranged in teams, with a captain for each team chosen from the team members. Prove that $\sum_{n \geq 0} t_n x^n / n! = e^{(xe^x)}$.

4. Given that n is prime, count the distinguishable n -bead necklaces that can be formed when k colors of beads are available.

PART II

5. Prove that a graph $G = (V, E)$ with at least 5 vertices is 4-connected if and only if for arbitrary five vertices $x, y, u, v, z \in V$, there exists an x, y -path that passes through u and avoids v and z .

6. State and prove Hall's Theorem.

7. Let G be a connected 5-regular graph with exactly one cut edge xy . Let G_1 and G_2 be the components of $G - xy$. Prove that if G_1 and G_2 are 4-edge-connected, then G has a perfect matching.

8. The *Kneser graph* $K(n, k)$ has vertex set $\binom{[n]}{k}$, with two vertices adjacent if they are disjoint k -sets. Prove that $\chi(K(n, k)) \leq n - 2k + 2$ by covering the vertices with $n - 2k + 2$ independent sets. Prove that

this is optimal when $n = 2k + 1$. (Example: The Petersen graph is $K(5, 2)$. Comment: In fact equality always holds.)

PART III

9. Let I be an order ideal in a rank-symmetric LYM order. Prove that the elements in I have average rank at most $n/2$, where n is the rank of the maximal elements.

10. Prove or disprove:

(a) Every graph G with chromatic number k has a proper k -coloring in which some color class has $\alpha(G)$ vertices, where $\alpha(G)$ denotes the independence number of G .

(b) If G is a connected graph, then the chromatic number of G is at most $1 + a(G)$, where $a(G)$ is the average of the vertex degrees of G . In other words, $a(G) = 2|E(G)|/|V(G)|$.

11. Prove that for large n with probability at least $1/2$, the maximum length of a constant consecutive string in a random list of n heads and tails is at least $0.9 \ln n$ and at most $1.1 \ln n$.

12. Explain how to construct a pair of orthogonal Latin squares of order 15. Include all needed building blocks but do not write out the final pair of matrices.