COMBINATORICS COMPREHENSIVE - Winter 2009

Submit *exactly* THREE problems from each Part. Passing requires good performance on each Part. Give full explanations, including CLEAR STATEMENTS of any theorems you use. Formulas without explanations do not receive much credit.

Part I

1. Place $2n$ points on a circle. Prove that the number of ways to pair up the points by drawing noncrossing chords inside the circle equals the number of ballot lists of length $2n$.

2. Let $\langle a \rangle$ satisfy $a_n = 3a_{n-1} - 2a_{n-2} + 2^n$ for $n \geq 2$, with $a_0 = a_1 = 1$. Express the generating function for $\langle a \rangle$ as a ratio of two polynomials. Using any method, obtain a formula for a_n as a function of n.

3. Evaluate $\sum_{k=1}^{n} (-1)^{k-1} k \binom{n}{k} 2^{n-k}$.

4. A mathematics department has n professors and $2n$ courses, two assigned to each professor each semester. If the courses are assigned randomly, what is the probability that no professor teaches the same two courses in the spring as in the fall?

Part II

5. For $r \geq 1$, let G be a graph of even order that does not have $K_{1,r+1}$ as an *induced* subgraph (this does *not* imply $\Delta(G) \leq r$). Prove that if G is r-connected, then G has a perfect matching.

6. Let G be a 2-connected graph such that the graph obtained by deleting any edge is not 2-connected. Prove that G has a vertex of degree 2. Conclude that if G has n vertices, and $n \geq 4$, then G has at most $2n - 4$ edges.

7. Prove that G is m-colorable if and only if the cartesian product $G \square K_m$ has an independent set of size $|V(G)|$.

8. A graph has no loops or multi-edges. Prove that every planar graph with at least four vertices has at least four vertices with degree less than 6. For each even value of n with $n \geq 8$, construct an n-vertex planar graph that has exactly four vertices with degree less than 6.

Part II

9. Let G be an Eulerian graph (no loops or multi-edges) with at least three vertices. Prove that G has at least three vertices with the same degree.

10. Recall that an order ideal in a poset is a set I such that $x < y$ and $y \in I$ imply $x \in I$. Let P be a rank-symmetric LYM poset with rank n (an example is $\mathbf{2}^{[n]}$). Let I be an order ideal in P. Prove that the average rank of the elements in I is at most $n/2$. Show that the conclusion may fail if P is not an LYM poset.

11. Let H be a graph. In the random graph model where every edge is generated with constant probability p, prove that almost every graph contains H as an induced subgraph.

12. Suppose that a $(4m-1, 2m-1, m-1)$-design exists. Prove that there is a Hadamard matrix of order $4m$.