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COMBINATORICS COMPREHENSIVE - Winter 2008

Submit *only* FIVE problems from Part I and THREE problems from Part II; 80 points possible. Passing requires good performance on each Part. Justify answers; GIVE CLEAR STATEMENTS of any theorems you use.

Part I

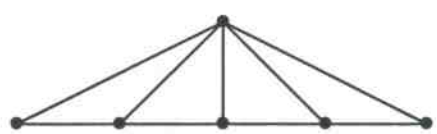
1. During $2n$ flips of a fair coin, a running total of heads and tails is kept. Compute the probability that the lead changes (one type leads and later the other type leads), given that each outcome appears n times.

2. For $n \geq 1$, let t_n be the number of spanning trees in the graph $K_1 \vee P_n$. (Note that $t_1 = 1$ and $t_2 = 3$.)

a) Obtain a many-term recurrence for $\langle t \rangle$.

b) Obtain a second-order recurrence for $\langle t \rangle$.

c) Prove inductively that $t_n = F_{2n}$, where F_k is the kth classical Fibonacci number ($F_0 = 0$ and $F_1 = 1$).



3. Given fixed positive integers $s_1, \ldots, s_n$, let $a_{n,k}$ be the number of integer solutions to $z_1 + \cdots + z_n = k$ such that $0 \leq z_i < s_i$ for each i.

a) Obtain the generating function $\sum a_{n,k} x^k$ (as a ratio of polynomials).

b) Without using (a), compute $a_{n,k}$ as a finite sum.

c) Explain why the answers to (a) and (b) are equivalent.

4. Given that n is an odd prime, count the distinguishable n-bead necklaces that can be formed when k colors of beads are available.

5. A cycle-factor of a digraph is a set of (directed) cycles such that each vertex lies on exactly one of the cycles. Prove that a digraph D has a cycle-factor if and only if $|N^+(S)| \geq |S|$ for all $S \subseteq V(D)$.

6. For each bound on the chromatic number given below, prove or disprove the statement that it holds for every n-vertex graph G.

a) $\chi(G) \leq \omega(G) + n/\alpha(G)$.

b) $\chi(G) \geq n/[n - \delta(G)]$.

7. Prove that there is no Eulerian plane graph in which one face has length 4 and the remaining faces have length 3. (Hint: Consider the dual graph.)

Part II

8. The *complete digraph* D_n with vertex set $[n]$ has $n(n-1)$ edges; each ordered pair of distinct vertices is an edge. A *monotone tournament* is a tournament in which every edge points toward its larger-indexed endpoint or every edge points toward its smaller-indexed endpoint. Given k, prove that if n is sufficiently large, then every subdigraph of D_n contains k vertices that form an independent set or induce a monotone tournament or induce a copy of D_k.

9. Recall that an order ideal in a poset is a set I such that $x < y$ and $y \in I$ imply $x \in I$. Let P be a rank-symmetric LYM poset with rank n (an example is $\underline{2}^{[n]}$). Let I be an order ideal in P. Prove that the average rank of the elements in I is at most $n/2$. Show that the conclusion may fail if P is not an LYM poset.

10. Let H be a hypergraph with m edges in which every edge has at least k vertices. Recall that if $m < 2^{k-1}$, then H is 2-colorable. Regardless of the value of m, prove that if each edge intersects fewer than 2^{k-3} other edges, then H is 2-colorable.

11. Explain how to construct a pair of orthogonal latin squares of order 15. Include all needed building blocks, but do not write out the final pair of squares.