COMBINATORICS COMPREHENSIVE - Winter 2007

Submit *only* FIVE problems from Part I and THREE problems from Part II; 80 points possible. Passing requires good performance on each Part. Justify answers; GIVE CLEAR STATEMENTS of any theorems you use.

Part I

1. *Families of subsets.*
a) Count the lists of subsets $A_0, A_1 \dots A_n$ of $[n]$ such that $A_0 \subset A_1 \subset \cdots \subset A_n$.
b) Count the lists of subsets $A_0, A_1 \dots A_n$ of $[n]$ such that $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n$.

2. Let $a_{n,k}$ be the number of compositions of n with k parts. Let $p_{n,k}$ be the number of partitions of n with k parts. (Compositions and partitions are different; in one model the order of the parts matters.)
a) Derive recurrences in two variables for both $a_{n,k}$ and $p_{n,k}$.
b) Obtain generating functions $A_3(x)$ and $P_3(x)$ for the compositions and for the partitions of integers into 3 parts. (Don't try to do these by solving the recurrences from part (a).)

3. Count the distinguishable ways to seat the people in n married couples at a rotating table with $2n$ seats so that no person sits next to his or her spouse. The sexes need not alternate. (Leave the answer as a summation.)

4. A financial company is designing a new symbol: a regular tetrahedron built from six metal bars of equal length. Given that the bars can be copper, silver, or gold, determine the number of distinguishable ways there are to do this (the tetrahedron can rotate in space but not reflect). How many ways are there using two bars of each type?

5. Suppose that $k \le m \le n$. Characterize the maximal subgraphs of $K_{m,n}$ that have no matching of size $k+1$. Which have the most edges?

6. Prove that $\chi(G) \ge n/[n - \delta(G)]$ for every n-vertex graph G. Prove that there are infinitely many graphs (other than complete graphs) where equality holds.

7. Give two proofs that the Petersen graph is nonplanar: one using a characterization of planar graphs, and one using Euler's Formula and girth.

Part II

8. Prove that the graph Ramsey number $R(mK_2, mK_2)$ equals $3m-1$.

9. The poset of subsets of $[n]$, ordered by inclusion, is an LYM order. Prove that every maximum antichain in this poset consists of a single rank.

10. Given natural numbers n and t, let $m = n - \binom{n}{t}^2 2^{1-t^2}$. Prove that there is a 2-coloring of the edges of $K_{m,m}$ with no monochromatic copy of $K_{t,t}$.

11. Prove that a subgraph of $K_{n,n}$ with no 4-cycle has at most $n(1+\sqrt{4n-3})/2$ edges. Prove that there exists such a subgraph with exactly this many edges for infinitely many values of n.