COMBINATORICS COMPREHENSIVE - Winter 2006

Do FIVE problems from Part I and THREE problems from Part II. Passing requires adequate performance on each Part. Answers must be justified in words; give clear statements of any theorems you use.

Part I

1. Let $\hat{F}_n$ be the nth adjusted Fibonacci number, defined by $\hat{F}_0 = \hat{F}_1 = 1$ and $\hat{F}_n = \hat{F}_{n-1} + \hat{F}_{n-2}$ for $n \geq 2$. Prove by induction and by combinatorial argument that $\hat{F}_n = \sum_{i=0}^{n} \binom{n-i}{i}$. (Use a combinatorial problem solved by this sequence).

2. Use generating functions to evaluate the sum $\displaystyle\sum_{k=0}^{r}(-1)^k\binom{n}{k}\binom{n}{r-k}$.

3. A department has n professors and $2n$ courses, two assigned to each professor each semester. How many ways are there to assign courses each fall? How many ways are there to assign them in the spring so no professor teaches the same two courses in the spring as in the fall?

4. A cycle-factor of a digraph is a set of (directed) cycles such that each vertex lies on exactly one of the cycles. Prove that a digraph D has a cycle-factor if and only if $|N^+(S)| \geq |S|$ for all $S \subseteq V(D)$.

5. Let G be a graph in which any two odd cycles have a common vertex. Prove that $\chi(G) \leq 5$. Construct a graph to show that the bound cannot be improved.

6. Prove that every simple planar graph with at least four vertices has at least four vertices with degree less than 6. For each even value of n with $n \geq 8$, construct an n-vertex simple planar graph that has exactly four vertices with degree less than 6.

Part II

7. In a simple digraph, the edge set is a subset of the set of ordered pairs of distinct vertices. A *monotone tournament* is a tournament in which all edges point toward the larger-indexed vertex or all edges point toward the smaller-indexed vertex. A complete digraph has each ordered pair of distinct vertices as an edge. Given m, prove that if N is sufficiently large, then every simple digraph with vertex set $[N]$ has an independent set of order m or a monotone tournament of order m or a complete loopless digraph of order m.

8. Let P be a rank-symmetric LYM poset with rank n (an example is $\underline{2}^{[n]}$). Let I be an order ideal in P. Prove that the average rank of the elements in I is at most $n/2$.

9. An army of computers is configured as a complete k-ary tree with leaves at distance l from the root (each nonleaf vertex has k "children" farther from the root). At a fixed time, each node works with probability p, independently of others. When a node is not working, it and the subtree below it are inaccessible. What is the expected number of nodes accessible from the root?

10. Fix $s, t, \in \mathbb{N}$. Say that a tournament T is s,t-*good* if for every choice of disjoint vertex sets S and T of sizes s and t, there is a vertex v such that $S \subseteq N^+(v)$ and $T \subseteq N^-(v)$. Prove that almost every tournament is s,t-good.