math 580

U. of Illinois COMBINATORICS COMPREHENSIVE EXAM – Fall 2012

Submit exactly THREE problems from each Part. Passing requires good performance on each Part. Give full explanations, including CLEAR STATEMENTS of any theorems that you use. Formulas without explanations do not receive much credit.

No external assistance permitted.

PART I

1. Use known identities to simplify and evaluate

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{n+1-k}$$

2. A *standard Young tableau* of shape (n,n) is a bijective filling of the boxes of a Ferrers diagram of for the partition (n,n) by the numbers $1,2,\ldots,2n$ such that the labels increase along rows and columns. For example, if $n=3$, the five tableaux are:

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array}.$$

Prove that if $SYT(n,n)$ is the set of all such tableaux then

$$\#SYT(n,n)=C_n:=\frac{1}{n+1}\binom{2n}{n}.$$

3. By a combinatorial construction with Ferrers diagrams, prove the following identity of generating series:

$$\prod_{k\geq 1}(1+zx^k)=1+\sum_{t\geq 1}\frac{z^t x^{t(t+1)/2}}{\prod_{k=1}^{t}(1-x^k)}$$

4. Let the set A_n be the set of permutations on $\{1,2,\ldots,n\}$ such that in one line notation, $i\,i+1$ does not appear, for $1\leq i\leq n-1$. Let B_n be the set of permutations π on $\{1,2,\ldots,n\}$ such that $\pi(i)\neq i+1$ for $1\leq i\leq n-1$. Prove that $\#A_n=\#B_n$.

(For example, if $n=3$, then $A_n=\{132,213,321\}$ whereas $B_n=\{123,321,312\}$.)

PART II

5. Prove that a graph $G=(V,E)$ with at least 6 vertices is 3-connected if and only if for arbitrary disjoint subsets A,B of V with $|A|,|B|\geq 3$, there exist three fully disjoint paths from A to B.

6. Prove König-Egervary Theorem on vertex covers using Hall's Theorem.

7. Prove that each 3-regular graph with at most two cut edges has a perfect matching.

8. Present an example of a 4-connected graph that is not 2-linked. What about 5-connected graphs?

PART III

9. Prove that every set of n integers has a subset summing to a multiple of n.

10. Prove or disprove:

(a) Every graph G with chromatic number k has a proper k-coloring in which some color class has $\alpha(G)$ vertices, where $\alpha(G)$ denotes the independence number of G.

(b) If G is a connected graph, then the chromatic number of G is at most $1 + a(G)$, where $a(G)$ is the average of the vertex degrees of G. In other words, $a(G) = 2|E(G)|/|V(G)|$.

11. Present an example of a tree and an ordering of its vertices such that the greedy coloring of this tree according to this ordering needs 5 colors.

12. Suppose H is a hypergraph with m edges in which every edge has size at least k and intersects fewer than $2^{k-1}/m$ other edges. Prove that H is 2-colorable.