COMBINATORICS COMPREHENSIVE - Fall 2007

Submit *only* FIVE problems from Part I and THREE problems from Part II; 80 points possible. Passing requires good performance on both Parts. Justify answers; GIVE CLEAR STATEMENTS of any theorems you use.

Part I

1. By counting a set in two ways, prove that $\sum_{i=1}^{n} i(n-i) = \sum_{i=1}^{n} \binom{i}{2}$.

2. Let $\langle a \rangle$ satisfy $a_n = 3a_{n-1} - 2a_{n-2} + 2^n$ for $n \geq 2$, with $a_0 = a_1 = 1$. Express the generating function for $\langle a \rangle$ as a ratio of two polynomials. Obtain a formula for a_n as a function of n.

3. Use generating functions to evaluate the sum $\sum_{k=0}^{r} (-1)^k \binom{n}{k} \binom{n}{r-k}$.

4. Suppose we roll a six-sided die until each of the numbers one through five have appeared at least once. What is the probability that we succeed sometime during the first n rolls?

5. Let S be a set of permutations of $[n]$. Prove that if $|S| \leq n/2$, then some permutation of $[n]$ differs in every position from every member of S. (Hint: Model this using a graph problem.)

6. Let G be a k-regular graph with connectivity 1. Determine $\chi'(G)$.

7. Use Euler's Formula to count the regions determined by a configuration of n lines in the plane when every two lines cross but no three lines have a common point.

Part II

8. Prove that $2m-1$ is the minimum t such that every 2-coloring of $E(K_{t,t})$ has a monochromatic connected subgraph with $2m$ vertices.

9. Consider a red/blue-coloring of the edges of a complete graph with more than m^2 vertices. Suppose that the red graph has a transitive orientation. Prove that the coloring has a monochromatic complete subgraph of order $m+1$. (Hint: Use posets.)

10. Let G be a bipartite graph with n vertices in which every vertex is given a list of more than $\log_2 n$ usable colors. Prove that a proper coloring of G can be chosen from the lists.

11. Suppose that a $(4m-1, 2m-1, m-1)$-design exists. Prove that there is a Hadamard matrix of order $4m$.