COMBINATORICS COMPREHENSIVE - Fall 2006

Do FIVE problems from Part I and THREE from Part II. Passing requires good performance on each Part. Justify answers; give clear statements of any theorems you use.

Part I

1. Let a_n be the number of arrangements of $2n$ people in 2 rows of length n such that heights increase in each row and column. By establishing a bijection involving a set of known size, prove that $a_n = \frac{1}{n+1}\binom{2n}{n}$.

2. Let $a(d_1, \ldots, d_n)$ be the number of trees with vertex set $[n]$ in which for each i, the degree of i is d_i. Obtain a recurrence for $a(d_1, \ldots, d_n)$ and use it to prove that $a(d_1, \ldots, d_n) = \binom{n-2}{d_1-1, \ldots, d_n-1}$. Use this result to prove Cayley's Formula for the number of trees with vertex set $[n]$.

3. Use generating functions to evaluate the sum below. (Hint: It is easier without convolution.)

$$\sum_{k=1}^{n} (-1)^{n-k} k \binom{n}{k} 2^k.$$

4. A rotating square table has a pocket at each corner. In each pocket we have the choice to place 1, 2, or 3 marbles. Compute the total number of distinguishable arrangements. Explain how to use the pattern inventory to obtain the number of distinguishable arrangements with a total of 7 marbles.

5. Use Tutte's 1-factor Theorem to prove that every connected line graph of even order has a perfect matching. Interpret the result as a statement about decomposition of graphs into subgraphs.

6. Let G be a k-regular graph with connectivity 1. Determine $\chi'(G)$.

7. Let G be a 3-regular connected plane graph in which every vertex is incident to one face of length 4, one face of length 6, and one face of length 8. Without drawing G, count the faces of G.

Part II

8. Prove that every simple Eulerian graph with at least three vertices has at least three vertices with the same degree.

9. Consider a red/blue-coloring of the edges of a complete graph with more than m^2 vertices. Suppose that the red graph is transitively orientable. Prove that the coloring has a monochromatic complete subgraph of order $m+1$. (Hint: Use posets.)

10. Let G be a graph with m edges. Use the probabilistic method to prove that if G has a matching of size k, then G has a bipartite subgraph with at least $(m+k)/2$ edges.

11. Consider the random graph model $\mathbb{G}(n,p)$, where $p = o(1/n)$. Prove that in this model almost every graph has no cycles.