LOGIC COMPREHENSIVE EXAM, JANUARY 2014

Each problem is worth 20 points for a total of 100 points.

Problem 1. Let L be the first-order language that consists of just a unary relation symbol. Consider the L-structure $\mathcal{A} = (\mathbb{N}, E)$, where E is the set of even natural numbers. Let D be the set of natural numbers divisible by 3. Prove that D is not definable in $\mathcal{A}$, even when parameters are allowed. That is, show that for any L-formula $\varphi(x, y_1, ..., y_n)$ and any $k_1, ..., k_n \in \mathbb{N}$,

$$D \neq \{m \in \mathbb{N} \ : \ \mathcal{A} \models \varphi(m, k_1, ..., k_n)\}.$$

Problem 2. Let L be the language that consists of a 2-ary function symbol. Consider the L-structures $(\mathbb{Z}, +)$ and $(\mathbb{Z}\times\mathbb{Z}, +)$, where addition in the second structure is defined coordinate-wise. Find an L-sentence that is true in one structure, but not in the other.

Problem 3. Let L be a language with just a unary function symbol and let $\mathcal{A} := (A, f)$ be an L-structure such that f is a bijection on A. Suppose further that there is no positive integer n such that f^n is the identity. (Here $f^1 := f$ and $f^{n+1} := f \circ f^n$). Show that there is a countable L-structure $\mathcal{B} = (B, g)$ that satisfies the same L-sentences as $\mathcal{A}$ and there exists $b \in B$ such that $b, g(b), g^2(b), ...$ are all distinct.

Problem 4. Let L consists of just a unary function symbol F. Let Σ consist of the following infinitely many sentences:

$$\forall x\big[(\exists y Fy = x) \wedge (\forall y_1 \forall y_2((Fy_1 = x \wedge Fy_2 = x) \to y_1 = y_2))\big]$$

and for each $n \in \mathbb{N}$ with $n > 1$,

$$\exists x_1 ... \exists x_n \bigwedge_{i \neq j} x_i \neq x_j,$$

and for each $n \in \mathbb{N}$ with $n \geq 1$,

$$\forall x \ F^n x \neq x.$$

(Here $F^1 = F$ and $F^{n+1} := FF^n$.) Show that Σ has quantifier elimination.

Problem 5. Let $f : \mathbb{N} \to \mathbb{N}$ be a computable function, $g : \mathbb{N} \to \mathbb{N}$ be a injective computable function such that $g(\mathbb{N})$ is computable and $g(n) \leq f(n)$ for all $n \in \mathbb{N}$. Show that also $f(\mathbb{N})$ is computable.