570

LOGIC COMPREHENSIVE EXAM, JANUARY 2012

Each problem is worth 20 points for a total of 100 points.

1. Let the language L have no non-logical symbols. In each part below, either prove the given statement or show that there is a counterexample.

(i) (10 points) For all L-sentences σ, τ, if σ is satisfiable and τ is satisfiable, then $\sigma \wedge \tau$ is satisfiable.

(ii) (10 points) For all L-sentences σ, τ, if σ is logically valid and $\sigma \to \tau$ is satisfiable, then τ is logically valid.

2. Suppose that the only non-logical symbol in L is a unary predicate symbol P. Let Σ consist of the following sentences:

$$\exists v_1 \exists v_2 \cdots \exists v_k \big(\bigwedge_{1 \le i < j \le k} v_i \neq v_j \wedge \bigwedge_{1 \le i \le k} P(v_i) \big), \quad \text{for } k = 2, 3, \ldots$$

and

$$\exists v_1 \exists v_2 \cdots \exists v_k \big(\bigwedge_{1 \le i < j \le k} v_i \neq v_j \wedge \bigwedge_{1 \le i \le k} \neg P(v_i) \big), \quad \text{for } k = 2, 3, \ldots.$$

Show that Σ is complete.

3. Let P be a set of prime numbers. Show that there is a model M of $\mathrm{Th}(\mathbb{N}; 0, S, +, \cdot, <)$ and an element $b \in M$ such that for each prime number p we have

$$M \models \exists x \big((S^p 0) \cdot x = b \big) \text{ if and only if } p \in P.$$

4. Find all subsets of $\mathbb{Q}^2$ that are 0-definable in the structure $(\mathbb{Q}; <)$. (Note: the question is about subsets of $\mathbb{Q}^2$, not subsets of $\mathbb{Q}$.)

5. Let L contain the constant symbol 0 and the unary function symbol S, and let Σ be a set of L-sentences.

(i) (8 points) State what it means for a relation $R \subseteq \mathbb{N}^k$ to be representable in Σ.

(ii) (12 points) Suppose $A_1, A_2, \ldots, A_k$ are pairwise disjoint computably enumerable subsets of $\mathbb{N}$ with $\bigcup_{i=1}^{k} A_i = \mathbb{N}$. Show that A_i is computable for each i with $1 \le i \le k$.