Math 570

LOGIC COMPREHENSIVE EXAM, AUGUST 2012

Each problem is worth 20 points for a total of 100 points.

Throughout, L is a (first-order) language, and for a set Σ of L-sentences, $Mod(\Sigma)$ is the class of models of Σ.

Problem 1. Let Σ_1, Σ_2 be sets of L-sentences such that no L-structure is a model of both Σ_1 and Σ_2. Show that there is an L-sentence σ such that $Mod(\Sigma_1) \subseteq Mod(\sigma)$ and $Mod(\Sigma_2) \subseteq Mod(\neg\sigma)$.

Problem 2. Suppose σ and τ are L-sentences and no non-logical symbol occurs in both σ and τ. Suppose also that every model of σ is infinite and every model of $\neg\tau$ is infinite. Finally, suppose that $\sigma \to \tau$ is true in all L-structures. Prove that either $\neg\sigma$ is true in all L-structures or τ is true in all L-structures.

Problem 3. Suppose that L has just a unary relation symbol P and a binary relation symbol $<$. Let T be the theory whose models are the structures $\mathcal{A} = (A, P^{\mathcal{A}}, <^{\mathcal{A}})$, where $(A, <^{\mathcal{A}})$ is a dense linear order without endpoints and $P^{\mathcal{A}}$ is a non-empty subset of A such that whenever $b \in P^{\mathcal{A}}$ and $a <^{\mathcal{A}} b$, then $a \in P^{\mathcal{A}}$. Find all complete L-theories extending T by indicating for each such complete extension T' a sentence σ' such that $T \cup \{\sigma'\}$ axiomatizes T'.

Problem 4. Suppose the only non-logical symbol of L is a binary predicate R. Consider the L-structure

$$\mathcal{A} := (\mathbb{Z}, \{(a, b) \in \mathbb{Z}^2 \colon a^2 = b^2\}).$$

(a) For which $k \in \mathbb{Z}$ is the singleton $\{k\}$ 0-definable in $\mathcal{A}$?

(b) For which infinite cardinals κ is $\mathrm{Th}(\mathcal{A})$ κ-categorical? (To be done without using results in model theory beyond Math 570.)

Problem 5. Let L contain (at least) the constant symbol 0 and the unary function symbol S. Let Σ be a set of L-sentences.

(a) Define what it means for a function $f\colon \mathbb{N} \to \mathbb{N}$ to be representable (as a function) in Σ.

(b) Suppose that $f, g\colon \mathbb{N} \to \mathbb{N}$ are functions that are representable in Σ and $h = g \circ f$ is their composition. Show that h is representable in Σ.