Math 561, Spring 2011
Final, May 10

Do your best. We are as much interested in your ability to think and reason as the correct answers.

1. **50 points** Let $\{X_n;\ n \in \mathbb{N}\}$ be independent and identically distributed random variables with common law given by $\mathbb{P}\{X_1 = 1\} = p \in (0,1) \setminus \{\frac{1}{2}\}$ and $\mathbb{P}\{X_1 = -1\} = 1-p$. Define $S_n = \sum_{i=1}^n X_i$. For any integer x, define $T_x = \inf\{n : S_n = x\}$. Let $\varphi(x) = (\frac{1-p}{p})^x$. Fix also integers $a < 0 < b$ and define $T \overset{\text{def}}{=} \min\{T_a, T_b\}$.

   (a) **10 points** Show that $\varphi(S_n)$ is a martingale with respect to the filtration
   $$\mathscr{F}_n \overset{\text{def}}{=} \sigma\{X_m;\ m \le n\}.$$

   (b) **20 points** Show that $\mathbb{P}\{T < \infty\} = 1$. Hint: compute $\lim_{n\to\infty} \frac{1}{n} \ln \varphi(S_n)$.

   (c) **20 points** Show that
   $$\mathbb{P}\{T_a < T_b\} = \frac{\varphi(b) - \varphi(0)}{\varphi(b) - \varphi(a)}.$$
   Hint: Consider the quantity $\varphi(S_T)$.

2. **50 points** Let X be a bounded or nonnegative random variable, and let $\mathscr{G}$ be a sub sigma-algebra of $\mathscr{F}$. Let $\mathbb{P}'$ be a second probability measure on $(\Omega, \mathscr{F})$ which is absolutely continuous with respect to $\mathbb{P}$, and let $\mathbb{E}'$ be the expecation operator associated with $\mathbb{P}'$.

   (a) **10 points** Prove that $\mathbb{P}'$-a.s., $\mathbb{E}\left[\frac{d\mathbb{P}'}{d\mathbb{P}} \middle| \mathscr{G}\right] > 0$.

   (b) **40 points** Prove that
   $$\mathbb{E}'[X|\mathscr{G}] = \frac{\mathbb{E}\left[X\frac{d\mathbb{P}'}{d\mathbb{P}} \middle| \mathscr{G}\right]}{\mathbb{E}\left[\frac{d\mathbb{P}'}{d\mathbb{P}} \middle| \mathscr{G}\right]}.$$

3. **50 points** Let $\{\mu_i;\ i \in \mathscr{I}\}$ be a collection of probability measures on $\mathbb{R}$, where $\mathscr{I}$ is some index set. For each $i \in \mathscr{I}$, define the characteristic function
   $$\varphi_i(\theta) \overset{\text{def}}{=} \int_{\mathbb{R}} e^{\sqrt{-1}\theta x} \mu_i(dx). \qquad \theta \in \mathbb{R}$$
   Recall that $\{\mu_i;\ i \in \mathscr{I}\}$ is said to be tight if for every $\varepsilon > 0$, there is a compact subset K of $\mathbb{R}$ such that
   $$\sup_{i \in \mathscr{I}} \mu_i(K^c) < \varepsilon.$$
   Show that $\{\mu_i;\ i \in \mathscr{I}\}$ is tight if $\{\varphi_i;\ i \in \mathscr{I}\}$ is equicontinuous in a neighborhood of the origin. The point here is that control of the characteristic function near the origin controls the tail behavior of the μ_i's. We will break this up into several parts.

   (a) **10 points** First prove that
   $$\frac{1}{2\delta}\int_{-\delta}^{\delta}\left\{1 - \frac{\varphi_i(\theta) + \varphi_i(-\theta)}{2}\right\} d\theta = \int_{x \in \mathbb{R}\setminus\{0\}} \left\{1 - \frac{\sin(\delta x)}{\delta x}\right\} \mu_i(dx).$$

(b) 20 points Show that

$$\mu_i\left([-L,L]^c\right) \le \frac{L}{4}\int_{\theta=-2/L}^{2/L}\left\{1-\frac{\varphi_i(\theta)+\varphi_i(-\theta)}{2}\right\}d\theta.$$

Hint: you might first understand the structure of the function $f(x) \overset{\text{def}}{=} 1-\frac{\sin(x)}{x}$. You might separately consider the cases $|x|\ge 2$ and $|x|\le 2$.

(c) 20 points Show that indeed $\{\mu_i;\ i\in\mathscr{I}\}$ is tight if $\{\varphi_i;\ i\in\mathscr{I}\}$ is equicontinuous in a neighborhood of the origin.

4. 50 points Suppose that $\{X_1, X_2, \dots\}$ is an independent collection of random variables such that $\mathbb{E}[X_n]=0$ and $\mathbb{E}[X_n^4]\le 1$. Show that $\lim_{n\to\infty}\frac{S_n}{n}=0$ almost surely.