561

Probability Comprehensive Exam, Spring 2010

1. (20 points) Suppose that $r(k)$, $k = 0, 1, 2, \cdots$, is a sequence of positive numbers such that $r(k) \to 0$ as $k \to \infty$. Show that if X_n, $n = 1, 2, \cdots$, is a sequence of random variables such that $EX_n = 0$ and $E(X_n X_m) \le r(n-m)$ for all $n = 1, 2, \cdots$ and $m = 1, \cdots, n$, then

$$\frac{X_1 + \cdots + X_n}{n} \to 0$$

in L^2 and in probability.

2. (30 points) Let $X_1, X_2, \cdots$ be independent random variables with $EX_i = 0$ and $\mathrm{Var} X_i \le C < \infty$. Put $S_n = X_1 + \cdots + X_n$ and $D_n = \max_{n^2 \le k < (n+1)^2} |S_k - S_{n^2}|$. (a) Use the Borel-Cantelli lemma to show that

$$\lim_{n\to\infty} \frac{S_{n^2}}{n^2} = 0$$

almost surely. (b) Use the Borel-Cantelli lemma to show that

$$\lim_{n\to\infty} \frac{D_n}{n^2} = 0$$

almost surely. (c) Use the results of (a) and (b) to show that

$$\lim_{n\to\infty} \frac{S_n}{n} = 0$$

almost surely.

3. (20 points) (a) Suppose that $X_n \Rightarrow X$, $Y_n \ge 0$ and $Y_n \Rightarrow c$, where c is a positive constant. Show that $X_n Y_n \Rightarrow cX$. Here $\Rightarrow$ stands for convergence in distribution. (b) Suppose that $Z_1, Z_2, \cdots$ are independent and identically distributed random variables with $EZ_1 = 0$ and $E(Z_1^2) = \sigma^2 \in (0, \infty)$. Show that

$$\frac{\sum_{m=1}^n Z_m}{(\sum_{m=1}^n Z_m^2)^{1/2}}$$

converges in distribution to a standard normal random variable as $n \to \infty$.

4. (30 points) Suppose that $X_j, j \ge 1$, are independent and identically distributed random variables with $P(X_1 = 1) = P(X_1 = -1) = \frac{1}{2}$. Put $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \ge 1$. Let τ be the stopping time defined by

$$\tau(\omega) = \inf\{n \ge 0 : S_n(\omega) = -a, \text{ or } S_n(\omega) = b)\},$$

where a and b are positive integers. (i) Show that $S_n^2 - n$ is a martingale and find $P(S_\tau - b)$. (ii) Find the expectation of τ. (iii) Assume that $a = b$. Show that $S_n^4 - 6nS_n^2 + 3n^2 + 2n$ is a martingale and find the variance of τ.