561

Probability Comprehensive Exam, May 2009

1. (25 points) Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed positive random variables with $P(X_1 > x) = e^{-x}$ for all $x \geq 0$. Show that

$$\limsup_{n\to\infty} \frac{X_n - \ln n}{\ln \ln n} = 1$$

almost surely.

2. (25 points) Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables such that for some $\alpha \in (0, 2)$,

$$P(X_1 > x) = P(X_1 < -x), \quad P(|X_1| > x) = x^{-\alpha}, \quad x \geq 1.$$

Define $S_n = X_1 + \cdots + X_n$ for $n \geq 1$. Show that, as $n \to \infty$, $n^{-1/\alpha} S_n$ converges weakly to a symmetric α-stable random variable, that is a random variable with characteristic function $\phi(t) = \exp(-C|t|^\alpha)$ for some $C > 0$.

3. (20 points) Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{F}_n$, $n \geq 0$, be an increasing sequence of sub-σ-fields of $\mathcal{F}$. (a) Suppose that $H_n \geq 0$ is a sequence of bounded random variables such that H_n is measurable with respect to $\mathcal{F}_{n-1}$ for every $n \geq 1$. Show that, if X_n, $n \geq 0$, is a supermartingale with respect to $\{\mathcal{F}_n, n \geq 0\}$, then $(H \cdot X)_n$, $n \geq 1$, is a supermartingale with respect to $\{\mathcal{F}_n, n \geq 1\}$, where for each $n \geq 1$,

$$(H \cdot X)_n = \sum_{m=1}^{n} H_m (X_m - X_{m-1}).$$

(b) Suppose that X_n, $n \geq 0$, is a supermartingale with respect to $\{\mathcal{F}_n, n \geq 0\}$ and T is a stopping time with respect to $\{\mathcal{F}_n, n \geq 0\}$. Show that $X_{T\wedge n}$ is a supermartingale with respect to $\{\mathcal{F}_n, n \geq 0\}$.

4. (25 points) Suppose that Y_n is a sequence of real valued random variables converging weakly to a real-valued random variable Y, and that Z_n is a sequence of nonnegative real-valued random variable converging weakly to a constant $c > 0$. Show that $Y_n Z_n$ converges weakly to cY.