COMPREHENSIVE EXAM, MATH 561

1. (60 pts) Let X be an exponential random variable with parameter 1. For each $t \geq 0$ define

$$M_t = 1_{\{X \leq t\}} - X \wedge t, \quad \mathcal{F}_t = \sigma(\{X \leq s\} : s \leq t).$$

a) Show that M is adapted to $\{\mathcal{F}_t\}$.

b) For any $A \in \mathcal{F}_s$, $A \cap \{X > s\}$ is either $\emptyset$ or $\{X > s\}$. *Hint:* Use Dynkin's $\pi - \lambda$-theorem.

c) Show that M is a martingale.

2. (40 pts) Two common modes of convergence of random variables are convergence in probability and almost sure convergence. We say the *weak law* (resp. *strong law*) of large numbers holds for a sequence $X_1, X_2, \ldots$ of integrable, centered random variables if

$$\frac{X_1 + \cdots + X_n}{n} \to 0, \quad \text{as } n \to \infty,$$

in probability (resp. almost surely).

a) If the weak law of large numbers holds, show that then $X_n/n \to 0$ in probability.

b) If the strong law of large numbers holds, show that then $X_n/n \to 0$ almost surely.