Comprehensive Exam: Probability

Name:

Do as many problems as you can. We are particularly interested in your techniques.

1. (25 pts) Let $\{X_n\}_{n=0}^{\infty}$ be a sequence of real-valued random variables with $X_0 \equiv 0$ and such that for some $\theta \in (0,1)$

$$\sum_{n=0}^{\infty} \frac{1}{\theta^{2n}} \mathbb{E}[(X_{n+1} - X_n)^2] < \infty.$$

Show that $P\{\sup_{n\geq 0} |X_n| < \infty\} = 1$.

2. (*Kolmogorov's 0-1 law*) (25 pts) Suppose $X_1, X_2, \ldots$ are independent, $\mathcal{G}_n = \sigma(X_n, X_{n+1}, \ldots)$ (i.e. the σ-field generated by $X_n, X_{n+1}, \ldots$), $\mathcal{T} = \bigcap_n \mathcal{G}_n$, and $A \in \mathcal{T}$. Show that $P(A)$ is either 0 or 1.

3. (*Variation norm*) (50 pts) For two probability measures μ and ν on $\mathbb{R}$ with its Borel σ-field put

$$\rho(\mu, \nu) = \sup_{\|\varphi\| \leq 1} \left| \int \varphi \, d\mu - \int \varphi \, d\nu \right|,$$

where the supremum is taken over all measurable functions φ with $\sup_{x\in\mathbb{R}} |\varphi(x)| \leq 1$.

(a) Show that ρ defines a metric and that $\rho(\mu_n, \mu) \to 0$ as $n \to \infty$ implies weak convergence of the sequence $\{\mu_n\}$ to μ.

(b) If μ and ν are atomic and attribute weights p_k and q_k to the point a_k, show that then $\rho(\mu, \nu) = \sum_k |p_k - q_k|$.

(c) Let X_1, X_2, Y_1, Y_2 be random variables on a probability space $(\Omega, \mathcal{F}, P)$, with respective distributions $\mu_1, \mu_2, \nu_1, \nu_2$. Assume that all variables take values in $\{0, 1, 2, \ldots\}$, that X_1, X_2 are independent, that Y_1, Y_2 are independent, and denote μ the distribution of $X_1 + X_2$, and ν the distribution of $Y_1 + Y_2$. Show that

$$\rho(\mu, \nu) \leq \rho(\mu_1, \nu_1) + \rho(\mu_2, \nu_2).$$

(Hint: You may find the equality $ad - bc = \frac{1}{2}[(a-b)(c+d) + (d-c)(a+b)]$ useful.)

(d) Let μ attribute weight p to the point 1 and $q = 1 - p$ to the point 0. If ν is the Poisson distribution with mean p (i.e. $\nu(\{n\}) = \frac{p^n}{n!} e^{-p}$, $n = 0, 1, 2, \ldots$), show that $\rho(\mu, \nu) \leq 2p^2$.

(e) (*Approximation by the Poisson distribution*) Suppose μ is the distribution of the number of successes in n Bernoulli trials with probabilities $p_1, \ldots, p_n$, and ν is the Poisson distribution with expectation $p_1 + \cdots + p_n$. Show that $\rho(\mu, \nu) \leq 2(p_1^2 + \cdots + p_n^2)$. Conclude that if for each n, $p_1 = \cdots = p_n = \lambda/n$, then the number of successes converges weakly to a Poisson random variable with mean λ.