Comprehensive exam, Probability, May 2006.

1. (25 pts) For probability measures μ, ν on $\mathbb{R}$, define

$$\|\mu - \nu\|_{\text{var}} = \sup\{|\int \phi \, d\mu - \int \phi \, d\nu|\},$$

where the supremum ranges over all measurable functions ϕ with $|\phi(x)| \leq 1$ for all $x \in \mathbb{R}$.

a) Show that this defines a metric on the space of probability measures.

b) If μ and ν are absolutely continuous with respect to the probability measure λ (for example $\lambda = (\mu + \nu)/2$), show that

$$\|\mu - \nu\|_{\text{var}} = \int |f - g| \, d\lambda, \quad \text{where } f = \frac{d\mu}{d\lambda} \text{ and } g = \frac{d\nu}{d\lambda},$$

and conclude that $\|\mu - \nu\|_{\text{var}} \leq 2$, with equality if and only if there is a measurable set $A \subset \mathbb{R}$ such that $\mu(A) = 1$, and $\nu(A) = 0$.

c) If $\{\mu_n\}_{n=1}^{\infty}$ is a Cauchy sequence in this metric, show that there exists a probability measure μ with $\|\mu_n - \mu\|_{\text{var}} \to 0$ as $n \to \infty$. *Hint: Consider the probability measure* $\lambda = \sum_{n=1}^{\infty} \frac{\mu_n}{2^n}$ *and use part b).*

2. (25 pts) Let $\{X_n\}$ and $\{Y_n\}$ be sequences of random variables. If X_n converges weakly to a random variable X, and $Y_n - X_n$ converges to 0 in probability, show that then also Y_n converges weakly to X.

3. (25 pts) Let $X_1, X_2, X_3, \ldots$ be i.i.d. random variables with mean 0 and variance 1. Define $S_n = \sum_{k=1}^{n} X_k$. Let τ be a stopping time with respect to the filtration generated by $\{X_k\}$ such that $\mathbb{E}[\tau] < \infty$.

a) Show $\mathbb{E}[S_{\tau \wedge n}^2] = \mathbb{E}[S_{\tau \wedge (n-1)}^2] + P(\tau \geq n)$.

(b) Show that $S_{\tau \wedge n}^2$ is a Cauchy sequence in L^1.

(c) Conclude $\mathbb{E}[S_\tau^2] = \mathbb{E}[\tau]$.

4. (25 pts) Let $\{A_n\}_{n=1}^{\infty}$ be independent events. Assume that $\sum_{n=1}^{\infty} P(A_n) = \infty$. Show that

$$\lim_{n \to \infty} \frac{\sum_{k=1}^{n} 1_{A_k}}{\sum_{k=1}^{n} P(A_k)} = 1 \quad \text{in probability,}$$

where 1_{A_k} denotes the indicator of the event A_k. *Hint: Use Chebyshev's inequality on* $\sum_{k=1}^{n} 1_{A_k}$.