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Comprehensive Exam, Probability, January 2008

1. Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of random variables on a common probability space so that the variances are bounded, and so that $Cov[X_j, X_k] \to 0$ uniformly as $|j-k| \to \infty$. Prove that for every $\epsilon > 0$

$$\lim_{n\to\infty} P\left(\frac{1}{n}|\sum_{k=1}^{n}(X_k - \mathrm{E}[X_k])| \geq \epsilon\right) = 0.$$

2. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables on a common probability space with mean-value 0 and variance 1, and set $S_n = X_1 + \cdots + X_n$.

(a) Use the central limit theorem and Kolmogorov's 0−1-law to conclude that $\limsup S_n/\sqrt{n} = \infty$ almost surely.

(b) Use an argument by contradiction to show that $S_n/\sqrt{n}$ does not converge in probability.

3. Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of independent and identically distributed Gaussian random variables with mean-value 0 and variance 1. For each $n = 1, 2, \ldots$ set

$$M_n = \exp\left((\sum_{k=1}^{n} X_k) - n/2\right), \quad \mathcal{F}_n = \sigma(X_k, 1 \leq k \leq n).$$

a) Show that M is a martingale with respect to the filtration $\{\mathcal{F}_n\}$.

b) Find $\langle M \rangle$, the unique nondecreasing process such that $\langle M \rangle_1 = 0$, $\langle M \rangle_n$ is $\mathcal{F}_{n-1}$-measurable for $n > 1$, and $M_n^2 - \langle M \rangle_n$ is an $\mathcal{F}_n$-martingale.