5601

Probability Comprehensive Exam, January 2007

There are 3 questions, 80 points total. Partial credit will be given. All random variables are defined on a common probability triple $(\Omega, \mathscr{F}, \mathbb{P})$ with expectation operator $\mathbb{E}$. According to the departmental WWW page, the purpose of the exam is to ensure that you "have acquired a suitable mathematical foundation for undertaking high level research." Even if you can't fully complete all of the problems, we would like to see that you have enough of an understanding of the intellectual geography of probability theory, and enough of a mastery of techniques, that you could make some significant headway on these problems. Good luck.

1. $\boxed{\text{30 points}}$ (Durrett) Let $\{A_n\}_{n=1}^\infty$ be a countable collection of measurable sets.
   (a) $\boxed{\text{15 points}}$ Assume that $\lim_{n\to\infty} \mathbb{P}(A_n) = 0$ and $\sum_{n=1}^\infty \mathbb{P}(A_{n+1} \setminus A_n) < \infty$. Show that $\mathbb{P}(A_n \text{ i.o.}) = 0$.
   (b) $\boxed{\text{15 points}}$ Assume that the A_n's are independent, $\mathbb{P}(A_n) < 1$ for all n, and that $\mathbb{P}(\cup_{n=1}^\infty A_n) = 1$. Then $\mathbb{P}(A_n \text{ i.o.}) = 1$.

2. $\boxed{\text{30 points}}$ One of the important uses of probability is to bring *order* out of chaos; i.e., to identify limits when there is a lot of randomness. Thus the importance of asymptotics, which is the subject of this question. Define $f(t) \overset{\text{def}}{=} 3 + 20\cos t - 79e^{-2t}$ for all $t \in \mathbb{R}$. For each $\lambda > 0$, let ξ_λ be an exponential random variable with parameter λ; i.e.,

$$\mathbb{P}\{\xi_\lambda \le t\} = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \ge 0 \\ 0 & \text{if } t < 0 \end{cases}$$

   (a) $\boxed{\text{15 points}}$ Compute $\lim_{\lambda\to\infty} \mathbb{E}[f(\xi_\lambda)]$.
   (b) $\boxed{\text{15 points}}$ Compute $\lim_{\lambda\to 0} \mathbb{E}[f(\xi_\lambda)]$ (hint: you may want to integrate by parts).

3. $\boxed{\text{20 points}}$ Let X be a discrete-time submartingale with respect to a filtration $\{\mathscr{F}_n\}_{n=1}^\infty$. Give a proof of Doob's inequality; for any positive integer N and any $\alpha > 0$,

$$\mathbb{P}\left\{\max_{0\le n\le N} X_n \ge \alpha\right\} \le \frac{1}{\alpha}\mathbb{E}\left[X_N \chi_{\{\max_{0\le n\le N} X_n \ge \alpha\}}\right].$$

(hint: consider the events $A_n \overset{\text{def}}{=} \{X_n \ge \alpha, \max_{0\le m<n} X_m < \alpha\}$). Be clear in your arguments.