Math 561

Comprehensive Exam
Fall 2012

Do your best. We are as much interested in your ability to think and reason as the correct answers.

1. [40 points] Let $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ be a filtration of sub-sigma-algebras of $\mathscr{F}$. Recall that for any stopping time τ,

$$\mathscr{F}_\tau \overset{\text{def}}{=} \{A \in \mathscr{F} : A \cap \{\tau \le n\} \in \mathscr{F}_n \text{ for all } n \in \mathbb{N}\}.$$

   (a) [20 points] Suppose that τ_1 and τ_2 are two stopping times such that $\tau_1 \le \tau_2$. Show that $\mathscr{F}_{\tau_1} \subset \mathscr{F}_{\tau_2}$.

   (b) [20 points] Show that τ is $\mathscr{F}_\tau$ measurable.

2. [50 points] Let's construct the Prohorov metric on the collection $\mathscr{P}(\mathbb{R})$ of Borel probability measures on $\mathbb{R}$. Let $\mathscr{C}$ be the collection of closed subsets of $\mathbb{R}$, and for any subset A of $\mathbb{R}$ and any $\varepsilon > 0$, define

$$A^\varepsilon \overset{\text{def}}{=} \{x \in \mathbb{R} : \operatorname{dist}(x, A) < \varepsilon\},$$

where $\operatorname{dist}(x, A) \overset{\text{def}}{=} \inf_{y\in A} |x - y|$ for all $x \in \mathbb{R}$. For any μ and ν in $\mathscr{P}(\mathbb{R})$, define

$$\rho(\mu, \nu) \overset{\text{def}}{=} \inf\{\varepsilon > 0 : \mu(F) \le \nu(F^\varepsilon) + \varepsilon \text{ for all } F \in \mathscr{C}\}.$$

Fix two points x and y in $\mathbb{R}$.

   (a) [25 points] Directly show that $\rho(\delta_x, \delta_y) \le |x - y|$.

   (b) [25 points] Directly show that $|x - y| \le \rho(\delta_x, \delta_y)$.