Probability Comprehensive Exam, August 2006

There are 3 questions, 90 points total. Partial credit will be given. All random variables are defined on a common probability triple $(\Omega, \mathscr{F}, \mathbb{P})$ with expectation operator $\mathbb{E}$. According to the departmental WWW page, the purpose of the exam is to ensure that you "have acquired a suitable mathematical foundation for undertaking high level research." Even if you can't fully complete all of the problems, we would like to see that you have enough of an understanding of the intellectual geography of probability theory, and enough of a mastery of techniques, that you could make some significant headway on these problems. Good luck.

1. **20 points** (Durrett) This should be a fairly easy probabilistic question. For each $\varepsilon \in (0,1)$, let $\{\xi_k^\varepsilon\}_{k=1}^\infty$ be an i.i.d. collection of variables with

$$\mathbb{P}\{\xi_k^\varepsilon = \varepsilon\} = \mathbb{P}\{\xi_k^\varepsilon = -\varepsilon\} = \frac{1}{2}.$$

and let N_ε be independent of the ξ_n^ε's and be Poisson λ/ε^2. Define

$$X_\varepsilon \overset{\text{def}}{=} \sum_{k=1}^{N_\varepsilon} \xi_k^\varepsilon.$$

   (a) **10 points** Compute $\mathbb{E}[\exp[i\theta X_\varepsilon]]$.

   (b) **10 points** Compute $\lim_{\varepsilon \searrow 0} \mathbb{E}[\exp[i\theta X_\varepsilon]]$. What does this imply?

2. **30 points** This is a probabilistic question with a very analytical focus. Often probabilistic bounds require a significant amount of analysis. Let ξ be a random variable. Assume that the moment generating function

$$\varphi(\theta) \overset{\text{def}}{=} \mathbb{E}[\exp[\theta\xi]]$$

is finite in some interval $[a,b]$. Let $[a',b'] \subset (a,b)$. Here we will rigorously show that φ is differentiable on $[a',b']$ and that its derivatives are the appropriate moments.

   (a) **10 points** Show that $\sup_{\theta\in[a',b']} \mathbb{E}[|\xi|^n \exp[\theta\xi]] < \infty$.

   (b) **10 points** Fix $\theta^* \in [a',b']$. Write the Taylor expansion of order n of e^θ about θ^* with remainder. Easy points!

   (c) **10 points** Show that for $\theta^* \in [a',b']$,

$$\varphi(\theta) = \sum_{k=0}^{n} \frac{(\theta-\theta^*)^k \mathbb{E}[\xi^k \exp[\theta^*\xi]]}{k!} + (\theta-\theta^*)^{n+1} G(\theta,\theta^*)$$

   for all $\theta \in [a',b']$, where $\sup_{\theta\in[a',b']} G(\theta,\theta^*)$ is bounded.

3. **40 points** (Durrett) Now let's do a martingale problem. Suppose that $\{\xi_k\}_{k=1}^\infty$ are i.i.d. with common law μ. Suppose that μ is not a Dirac measure. Suppose furthermore that

$$\varphi(\theta) \overset{\text{def}}{=} \int_{z\in\mathbb{R}} e^{\theta z}\mu(dz)$$

is finite for $\theta \in (-1, 1)$, and define

$$\psi(\theta) \stackrel{\text{def}}{=} \ln \varphi(\theta)$$

for $\theta \in (-1, 1)$. Define $S_n \stackrel{\text{def}}{=} \sum_{1 \leq k \leq n} \xi_k$ for all $n \geq 0$. Fix $\theta \in (-1, 1)$ and define

$$X_n^\theta \stackrel{\text{def}}{=} \exp\left[\theta S_n - n\psi(\theta)\right]$$

for all $n \geq 0$.

(a) [10 points] Show that X^θ is a martingale.

(b) [10 points] Show that ψ is strictly convex (you recall Question 2).

(c) [10 points] Show that $\lim_{n\to\infty} \mathbb{E}\left[\sqrt{X_n^\theta}\right] = 0$.

(d) [10 points] Show that $\lim_{n\to\infty} X_n^\theta = 0$ $\mathbb{P}$-a.s. (Hint: you might use a (super/sub) martingale convergence theorem).