Comprehensive Exam PDEs Math 553 May 2018

Total points 100. Do 4 out of the 5 problems.

Instructions. Show all your work and make your explanations as full as possible. Calculators are not allowed on this exam, and neither are books or notes.

Problem 1 (25 points)

Let Ω be a bounded domain in $\mathbb{R}^n$. Let $\vec{V}(x,t)$ be a smooth vector field and consider the parabolic partial differential equation

$$\begin{cases} u_t - \Delta u + \vec{V} \cdot \nabla u = 0, & x \in \Omega, \quad t \in (0,T) \\ u(x,0) = u_0(x), & x \in \bar{\Omega} \\ u(x,t) = f(x,t), & x \in \partial\Omega \quad t \in (0,T). \end{cases} \tag{1}$$

a) (15 points) State and prove a weak maximum principle for the problem.

b) (10 points) Prove that there exists at most one smooth solution, continuous up to the boundary for the above problem.

Problem 2 (25 points)

Consider the conservation law $G'(u)u_x + u_t = 0$ for $(x,t) \in (a,b) \times (0,\infty)$ where $G \in C^1(\mathbb{R})$.

a) (5 points) Define an integral solution of the conservation law for $a \leq x \leq b$.

b) (5 points) Derive the jump (Rankine-Hugoniot) condition satisfied by a piecewise smooth integral solution u across a smooth curve where the solution has a discontinuity.

c) (15 points) Solve the conservation law when $G(u) = u^2 + u$ with initial condition

$$h(x) = u(x,0) = \begin{cases} 1 & \text{for } x < 0, \\ 0 & \text{for } x > 0. \end{cases}$$

Problem 3 (25 points)

In what follows we define the Fourier transform for appropriately smooth and decaying functions:

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} f(x)\, e^{-ix\cdot\xi}\, dx$$

and

$$f(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} \widehat{f}(\xi)\, e^{ix\cdot\xi}\, d\xi.$$

Recall also that

$$\widehat{f * g}(\xi) = (2\pi)^{\frac{1}{2}} \widehat{f}(\xi)\widehat{g}(\xi).$$

a) (5 points) Evaluate the Fourier transform of $\chi_{[-t,t]}(x)$ which is the function that equals 1 inside the interval $[-t,t]$ and zero otherwise.

b) (10 points) Using the Fourier transform method solve the initial value problem (IVP) for non-negative times $t \geq 0$

$$\begin{cases} u_{tt} + 2u_t - u_{xx} + u = 0, & x \in \mathbb{R}, \\ u(x,0) = 0, \quad u_t(x,0) = f(x), \end{cases} \tag{2}$$

where f is a smooth and compactly supported function.

Hint: *You may find the transformation $u(x,t) = e^{-t}v(x,t)$ useful.*

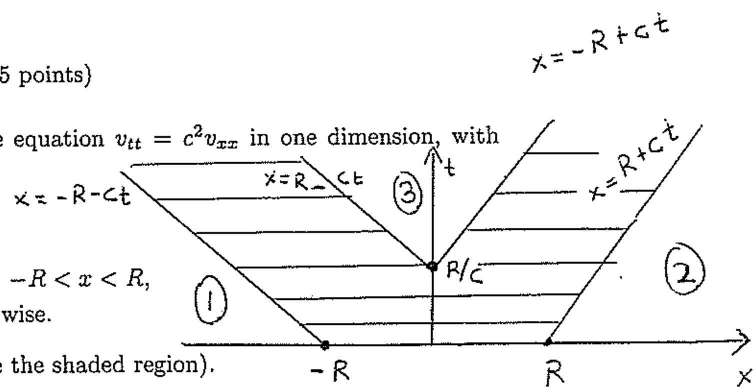
c) (10 points) Define an appropriate energy functional for the equation and show that the solution of the IVP of part b) is unique.

Problem 4 (25 points)

a) (5 points) Write down the solution of the wave equation $v_{tt} = c^2 v_{xx}$ in one dimension, with initial conditions $v(x,0) \equiv 0$ and $v_t(x,0) = \Psi(x)$.

b) (5 points) Let $R > 0$ and suppose

$$\Psi(x) = \begin{cases} x & \text{when } -R < x < R, \\ 0 & \text{otherwise.} \end{cases}$$



Show $v(x,t) = 0$ in Regions 1, 2, 3 (that is, outside the shaded region).

c) (10 points) Solve the wave equation $u_{tt} = c^2 \Delta u$ in three dimensions with radially symmetric initial conditions $u(\vec{x},0) \equiv 0$ and

$$u_t(\vec{x},0) = \psi(\vec{x}) = \begin{cases} 1 & \text{when } |\vec{x}| < R, \\ 0 & \text{otherwise.} \end{cases}$$

d) (5 points) Describe the region in three dimensions where u is supported (that is, where it is nonzero), at time $t = 2R/c$.

Problem 5 (25 points)

This problem is about certain properties of harmonic functions.

a) (7 points) Let Ω be a domain in $\mathbb{R}^n$ and $u \in C(\Omega)$ (continuous function). State (as an equation, not in words) what it means for u to satisfy the mean value property.

b) (18 points) Now suppose $\Omega = B_a(0) \subset \mathbb{R}^n$, the open ball of radius a centered at the origin. Define $\Omega_+ := \Omega \cap \mathbb{R}^n_+$ where $\mathbb{R}^n_+$ is the open half space and define $\Omega_0 := \{x \in \Omega \, : \, x_n = 0\}$. Let $u \in C^2(\Omega_+) \cap C(\Omega_+ \cup \Omega_0)$ be harmonic in Ω_+ with $u = 0$ on Ω_0. Show how to extend u to a harmonic function $\tilde{u}$ in all of Ω. Prove that your extension is indeed harmonic. Be sure to explain why the conditions on u stated above are needed.