Total points 100. Do 4 out of the 5 problems.

Instructions. Show all your work and make your explanations as full as possible. Calculators are not allowed on this exam, and neither are books or notes.

Problem 1 (25 points)

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \). Let \( \vec{V}(x, t) \) be a smooth vector field and consider the parabolic partial differential equation

\[
\begin{align*}
  u_t - \Delta u + \vec{V} \cdot \nabla u &= 0, & x \in \Omega, & t \in (0, T) \\
  u(x, 0) &= u_0(x), & x \in \overline{\Omega} \\
  u(x, t) &= f(x, t), & x \in \partial \Omega & t \in (0, T). 
\end{align*}
\]

(1)

a) (15 points) State and prove a weak maximum principle for the problem.

b) (10 points) Prove that there exists at most one smooth solution, continuous up to the boundary for the above problem.

Problem 2 (25 points)

Consider the conservation law \( G'(u)u_x + u_t = 0 \) for \( (x, t) \in (a, b) \times (0, \infty) \) where \( G \in C^1(\mathbb{R}) \).

a) (5 points) Define an integral solution of the conservation law for \( a \leq x \leq b \).

b) (5 points) Derive the jump (Rankine-Hugoniot) condition satisfied by a piecewise smooth integral solution \( u \) across a smooth curve where the solution has a discontinuity.

c) (15 points) Solve the conservation law when \( G(u) = u^2 + u \) with initial condition

\[
h(x) = u(x, 0) = \begin{cases} 
  1 & \text{for } x < 0, \\
  0 & \text{for } x > 0. 
\end{cases}
\]
Problem 3 (25 points)

In what follows we define the Fourier transform for appropriately smooth and decaying functions:

\[ \hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} f(x) e^{-ix\xi} \, dx \]

and

\[ f(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi} \, d\xi, \]

Recall also that

\[ \hat{f \ast g}(\xi) = (2\pi)^{\frac{1}{2}} \hat{f}(\xi)\hat{g}(\xi). \]

a) (5 points) Evaluate the Fourier transform of \( \chi_{[-t,t]}(x) \) which is the function that equals 1 inside the interval \([-t,t]\] and zero otherwise.

b) (10 points) Using the Fourier transform method solve the initial value problem (IVP) for non-negative times \( t \geq 0 \)

\[
\begin{cases}
  u_{tt} + 2u_t - u_{xx} + u = 0, & x \in \mathbb{R}, \\
  u(x,0) = 0, & u_t(x,0) = f(x),
\end{cases}
\]

where \( f \) is a smooth and compactly supported function.

Hint: You may find the transformation \( u(x,t) = e^{-t}v(x,t) \) useful.

c) (10 points) Define an appropriate energy functional for the equation and show that the solution of the IVP of part b) is unique.

Problem 4 (25 points)

a) (5 points) Write down the solution of the wave equation \( u_{tt} - c^2 u_{xx} \) in one dimension, with initial conditions \( u(x,0) \equiv 0 \) and \( u_t(x,0) = \psi(x) \).

b) (5 points) Let \( R > 0 \) and suppose

\[ \chi(x) = \begin{cases} 
  x & \text{when } -R < x < R, \\
  0 & \text{otherwise.}
\end{cases} \]

Show \( u(x,t) = 0 \) in Regions 1, 2, 3 (that is, outside the shaded region).

c) (10 points) Solve the wave equation \( u_{tt} = c^2 \Delta u \) in three dimensions with radially symmetric initial conditions \( u(\bar{x},0) \equiv 0 \) and

\[ u_t(\bar{x},0) = \psi(\bar{x}) = \begin{cases} 
  1 & \text{when } |\bar{x}| < R, \\
  0 & \text{otherwise.}
\end{cases} \]
d) (5 points) Describe the region in three dimensions where \( u \) is supported (that is, where it is nonzero), at time \( t = 2R/c \).

Problem 5 (25 points)

This problem is about certain properties of harmonic functions.

a) (7 points) Let \( \Omega \) be a domain in \( \mathbb{R}^n \) and \( u \in C(\Omega) \) (continuous function). State (as an equation, not in words) what it means for \( u \) to satisfy the mean value property.

b) (18 points) Now suppose \( \Omega = B_a(0) \subset \mathbb{R}^n \), the open ball of radius \( a \) centered at the origin. Define \( \Omega_+ := \Omega \cap \mathbb{R}^n_+ \) where \( \mathbb{R}^n_+ \) is the open half space and define \( \Omega_0 := \{ x \in \Omega : x_n = 0 \} \). Let \( u \in C^2(\Omega_+) \cap C(\Omega_+ \cup \Omega_0) \) be harmonic in \( \Omega_+ \) with \( u = 0 \) on \( \Omega_0 \). Show how to extend \( u \) to a harmonic function \( \bar{u} \) in all of \( \Omega \). Prove that your extension is indeed harmonic. Be sure to explain why the conditions on \( u \) stated above are needed.