Math 553 Exam

May, 2010

1. Let $\varphi \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \varphi(x)dx = 1$. For any $\varepsilon > 0$, define $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\varepsilon^{-1}x)$. Show that if $\varepsilon \to 0$ then $\varphi_\varepsilon \to \delta$ as distributions.
(Here δ is the delta distribution, and $\{\varphi_\varepsilon\}$ is called an approximation to the identity.)

2. Let $n \geq 3$. Show that $K(x) = \frac{1}{(2-n)\omega_n |x|^{n-2}}$ is the fundamental solution for the Laplace operator.

3. Let $\mathbf{E} = (E_1, E_2, E_3)$ and $\mathbf{H} = (H_1, H_2, H_3)$, where $E_1, E_2, E_3, H_1, H_2, H_3$ are C^2 functions of $(\mathbf{x}, t) \in \mathbb{R}^3 \times \mathbb{R}$. Suppose that $\mathbf{E}, \mathbf{H}$ satisfy Maxwell equation

$$\begin{cases} \mathbf{E}_t &= \operatorname{curl} \mathbf{H} \\ \mathbf{H}_t &= -\operatorname{curl} \mathbf{E} \\ \operatorname{div} \mathbf{E} &= \operatorname{div} \mathbf{H} = 0 \,. \end{cases}$$

Show that $E_1, E_2, E_3, H_1, H_2, H_3$ satisfy the 3-d wave equation $u_{tt} - \Delta u = 0$.

4. Let $u \in C^2(\mathbb{R} \times [0, \infty))$ solves the initial-value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x)\,, \end{cases}$$

where g, h are supported on the interval $[-R, R]$. The kinetic energy is

$$K(t) = \frac{1}{2}\int_{\mathbb{R}} u_t^2(x,t)dx$$

and the potential energy is

$$P(t) = \frac{1}{2}\int_{\mathbb{R}} u_x^2(x,t)dx\,.$$

Prove that
a) $K(t) + P(t)$ is constant in t.
b) (Equipartition of energy) $K(t) = P(t)$ for all large enough times t.

5. Consider the n-dimensional wave equation with dissipation

$$\begin{cases} u_{tt} - \Delta u + \alpha u_t = 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x)\,, \end{cases} \tag{1}$$

where g and h are supported on the ball $B(0, R)$ and $\alpha \geq 0$ is a constant. Show that if u is a solution of (1), then for fixed t, $u(\cdot, t)$ is a function with a compact support (whose size depends on R).

6. a) Use Fourier transform to derive a formula for the solution of Schrödinger's equation

$$\begin{cases} iu_t + \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty); \\ u(x,0) = g(x) & \text{for all } x \in \mathbb{R}^n. \end{cases} \tag{2}$$

Here u and g are complex-valued.

b) Use Part a) to show that if u is a solution of the Schrödinger equation (2), then

$$\|u(\cdot,t)\|_\infty \leq \frac{1}{(4\pi|t|)^{n/2}}\|g\|_1,$$

for each $t > 0$.