# Comprehensive Exam — PDEs (Math 553) — May 2009

**Total points** 100. Do **4** problems.

**Instructions** Show ALL your working and make your explanations as full as possible. Calculators are not allowed on this exam, and neither are books or notes.

Unless otherwise stated, $T > 0$ is fixed and $\Omega$ denotes a smoothly-bounded domain in $\mathbb{R}^n, n \geq 2$.

You may use Green's Formulas:

$$\int_\Omega [u\Delta v + \nabla u \cdot \nabla v] \, dx = \int_{\partial\Omega} u \frac{\partial v}{\partial \nu} \, dS$$

$$\int_\Omega [u\Delta v - v\Delta u] \, dx = \int_{\partial\Omega} \left[ u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right] dS$$

(1) (25 points) *(First order equations)* Consider the semilinear equation $xu_x + y^2 u_y = 3u$.

(a) Use the method of characteristics to solve the PDE using the Cauchy data $u(x, 1) = k(x)$, where $k$ is an arbitrary function.

(b) Explain how the method of characteristics can fail for Cauchy data of the form $x = s, y = g(s), z = h(s)$.

(2) (25 points) *(Laplace's equation)*

Let $u$ be a smooth function on $\mathbb{R}^n, n \geq 2$.

(a) Prove $u$ is harmonic if and only if

$$u(x) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u \, dS$$

for all $x \in \mathbb{R}^n, r > 0$. Here $|\partial B(x, r)|$ denotes the surface area of the sphere $\partial B(x, r)$.

(b) Briefly discuss the one dimensional case.

(3) (25 points) *(Eigenvalues of the Laplacian)* Let $u \not\equiv 0$ be a smooth function on $\overline{\Omega}$. Suppose $u$ is an eigenfunction of the negative Laplacian with eigenvalue $\lambda$, meaning $-\Delta u = \lambda u$ in $\Omega$ and $u = 0$ on $\partial\Omega$.

(a) Apply Green's Formula to prove $\lambda > 0$. (Justify your deductions.)

(b) Use the Maximum Principle to prove $\lambda > 0$. (Be careful to verify the hypotheses.)

(c) Explain the meaning of $\sqrt{\lambda}$ in terms of separation of variables and the wave equation.

(4) (25 points) *(Wave equation)*

(a) Solve the initial value problem for the wave equation in one dimension

$$\begin{cases} u_{tt} - u_{xx} = 0, & x \in \mathbb{R},\ t \in (0,\infty), \\ u(x,0) = f(x), u_t(x,0) = g(x), & x \in \mathbb{R}, \end{cases}$$

where $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$.

Do not use a solution formula. You should derive the solution from first principles.

(b) Apply your solution formula to $g(x) \equiv 0$ and

$$f(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x \geq 0. \end{cases}$$

(Obviously this $f$ is not $C^2$-smooth.) Evaluate $u(x,t)$ in the different regions of the $xt$-plane.

(c) Briefly explain how to show $u$ in part (b) is a weak solution of the wave equation. (*Hint:* transmission conditions.) You do not need to carry out the calculations.

(5) (25 points) *(Heat equation in bounded domain)* Assume $u(x,t)$ is smooth on $\overline{\Omega \times [0,T]}$ and solves:

$$\begin{aligned} u_t &= \Delta u, && x \in \Omega, \quad t > 0, \\ u(x,0) &= g(x), && x \in \Omega, \\ u(x,t) &= 0, && x \in \partial\Omega, \quad t > 0, \end{aligned}$$

where $g$ is smooth and $g \leq 0$.

(a) Explain why $u(x,t) \leq 0$ for all $x \in \Omega, 0 < t < T$.

(b) Suppose in addition $g$ has compact support in $\Omega$, and $g(x) < 0$ for some $x \in \Omega$. It can be shown $u(x,t) < 0$ for all $x \in \Omega, 0 < t < T$.

Use this fact to justify the claim that "the heat equation has infinite propagation speed".