Comprehensive Exam — PDEs (Math 553) — May 2008

Total points 100. Do **4** problems.

Instructions Show ALL your working and make your explanations as full as possible. Calculators are not allowed on this exam, and neither are books or notes.

You may use Green's Formulas:

$$\int_{\Omega} [u\Delta v + \nabla u \cdot \nabla v]\, dx = \int_{\partial\Omega} u \frac{\partial v}{\partial \nu}\, dS$$

$$\int_{\Omega} [u\Delta v - v\Delta u]\, dx = \int_{\partial\Omega} \left[u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right] dS$$

(1) (25 points) *(First order equation)*

Find the general solution of the equation

$$x u_{xx} + u_{xy} = 0$$

where $u(x, y)$ is a smooth function and $x, y \in \mathbb{R}$.

Hint. Let $v = u_x$.

(2) (25 points) *(Poisson's equation)*

(a) Show that $F(x) = -1/4\pi|x|$ is the fundamental solution of Laplace's equation in three dimensions:

$$\int_{\mathbb{R}^3} F(x)\Delta\phi(x)\, dx = \phi(0)$$

for every $\phi \in C_0^\infty(\mathbb{R}^3)$.

(b) Use F to write down a weak solution of Poisson's equation $\Delta u = g$ in $\mathbb{R}^3$. Then show formally that your u is a weak solution.

(3) (25 points) *(Wave equation)* For a continuous function $f(x)$ on $\mathbb{R}^n$, the spherical mean of f on a sphere of radius r and center x is

$$M_f(x,r) = \frac{1}{\omega_n}\int_{|\xi|=1} f(x+r\xi)\,dS_\xi$$

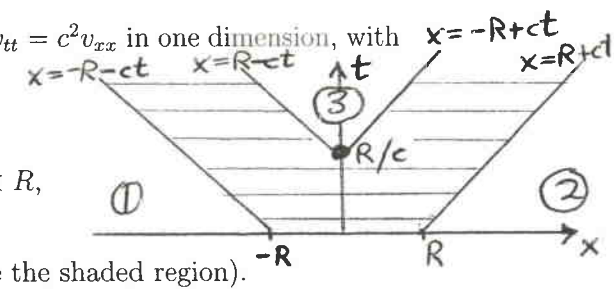
where ω_n denotes the surface area of the unit sphere in $\mathbb{R}^n$ and dS_ξ denotes the surface measure. Show that the spherical mean of f satisfies the Darboux equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right)M_f(x,r) = \Delta_x M_f(x,r).$$

Hint. Show first that $\frac{\partial}{\partial r}M_f(x,r) = \frac{1}{r^{n-1}}\Delta_x \int_0^r \rho^{n-1}M_f(x,\rho)\,d\rho$ by differentiating under the integral sign and applying the divergence theorem appropriately.

(4) (25 points) *(Wave equation)*

a) Write down the solution of the wave equation $v_{tt} = c^2 v_{xx}$ in one dimension, with initial conditions $v(x,0) \equiv 0$ and $v_t(x,0) = \Psi(x)$.



b) Let $R > 0$ and suppose

$$\Psi(x) = \begin{cases} x & \text{when } -R < x < R, \\ 0 & \text{otherwise.} \end{cases}$$

Show $v(x,t) = 0$ in Regions 1, 2, 3 (that is, outside the shaded region).

c) Solve the wave equation $u_{tt} = c^2\Delta u$ in three dimensions with radially symmetric initial conditions $u(\vec{x},0) \equiv 0$ and

$$u_t(\vec{x},0) = \psi(\vec{x}) = \begin{cases} 1 & \text{when } |\vec{x}| < R, \\ 0 & \text{otherwise.} \end{cases}$$

Hints. $v = ru$ and part (a).

d) Describe the region in three dimensions where u is supported (that is, where it is nonzero), at time $t = 2R/c$.

(5) (25 points) *(Heat equation)*

(a) Let $c \in \mathbb{R}$. Find an explicit solution formula for

$$\begin{cases} u_t - \Delta u + cu = 0, & x \in \mathbb{R}^n, \quad t \in (0, \infty), \\ u(x, 0) = f(x), \end{cases}$$

(b) Assume $u(x, t)$ is a smooth function solving

$$\begin{cases} u_t - u_{xx} - 2\gamma u_x = 0, & x \in (a, b), \quad t \in (0, \infty), \\ u(x, 0) = f(x), & x \in (a, b), \\ u(a, t) = u(b, t) = 0, & t \geq 0, \end{cases}$$

where $f \in C_0^\infty(a, b)$ and $\gamma \in \mathbb{R}$ is a constant.

Show that the energy

$$E(t) = \frac{1}{2} \int_a^b u(x, t)^2 \, dx$$

is dissipated. Hence show the solution u is unique.