Comprehensive Exam in PDE's - May 2007

The exam consists of 5 problems. Each problem is worth 25 points. You must do four out of five problems. You will receive credit for the best four. You must show all work to receive full credit.

Unless otherwise specified you may assume that all functions are well-behaved (C^∞ and rapidly decaying.) In general you may use any analytical facts you need as long as you state them clearly (and correctly).

Identities: You may find the following identities helpful:

$$\begin{aligned}
\cosh(2t) &= \cosh^2(t) + \sinh^2(t) \\
\sinh(2t) &= 2\sinh(t)\cosh(t) \\
2\int_0^1 \cos(n\pi x)\cos(m\pi x)dx &= \delta_{n,m} + \delta_{n,0}\delta_{m,0} \\
2\int_0^1 \sin(n\pi x)\sin(m\pi x)dx &= \delta_{n,m} - \delta_{n,0}\delta_{m,0}
\end{aligned}$$

Problem 1: Laplace Equation

The purpose of this problem is to establish that solutions of the Laplace equation satisfy the following minimization principle: that of all functions with finite Dirichlet energy satisfying given boundary data the one with the smallest Dirichlet energy is the one which satisfies the Laplace equation.

(i) Prove the above. Specifically suppose Ω is a domain in the plane with smooth boundary and $u, v \in C^2(\bar{\Omega})$, that u satisfies the Laplace equation with specified boundary data

$$\Delta u = 0 \qquad u|_{\partial\Omega} = f$$

and v is an arbitrary C^2 function that takes the same boundary data

$$v|_{\partial\Omega} = f$$

Then the following inequality holds

$$\int_\Omega |\nabla v|^2 \geq \int_\Omega |\nabla u|^2$$

with equality if and only if $u = v$.

Hint: It may help to write $v = u + w$. Note the boundary conditions on w.

(ii) Let Ω be the square $(0,1) \times (0,1)$. Suppose that u satisfies the Laplace equation on Ω:

$$\begin{aligned} \Delta u &= 0 \\ u(x,0) &= 0 \\ u(x,1) &= f(x) \\ u(0,y) &= 0 \\ u(1,y) &= 0 \end{aligned}$$

Compute the Dirichlet energy $\int |\nabla u|^2$ in terms of the Fourier Sine series components of $f(x)$.

(iii) For a function f given by a sine series

$$f(x) = \sum_{k=1}^{\infty} f_k \sin(\pi k x)$$

define the $\mathbf{H}_{\frac{1}{2}}$ norm as follows:

$$\|f\|^2_{\mathbf{H}_{\frac{1}{2}}} = \sum_{k=1}^{\infty} |k| f_k^2$$

use parts (i) and (ii) to show that if Ω is the unit square, and $v \in C^2(\bar{\Omega})$ satisfying

- $v(x,0) = 0 \quad v(0,y) = 0 \quad v(1,y) = 0$
- $v(x,1) = f(x)$

then

$$\int_\Omega |\nabla v|^2 \geq c\|f\|^2_{\mathbf{H}_{\frac{1}{2}}}$$

for suitable constant c.

This is a simple example of a restriction result: if v is a function in $\mathbf{H}_1(\mathbf{R}^2)$ then the restriction to the line $y = 1$ makes sense as a function in $\mathbf{H}_{\frac{1}{2}}(\mathbf{R})$. Note that there is a loss of half a derivative (in the Sobolev sense) when restricting v from the plane to the line.

Problem 2: Quasi-linear equation Consider the problem

$$u_t + (\frac{u^4}{4})_x = 0$$

with the initial data

$$u(x,0) = \begin{cases} 2 & x \leq -8 \\ -x^{\frac{1}{3}} & x \in (-8,0) \\ 0 & x \in [0,1) \\ -1 & x > 1 \end{cases}$$

Describe the solution for all times. In what sense does it satisfy the equation.

Problem 3: Duhamel Principle

(i) Suppose that the evolution

$$u_t = \mathcal{L}u \qquad u(x,0) = u_0(x)$$

has the solution

$$u(x,t) = \int K(x,x',t)u_0(x')dx',$$

where $\mathcal{L}$ is a time-independent linear operator. Find the solution to

$$u_t = \mathcal{L}u + f(x,t) \quad u(x,0) = 0$$

(ii) Find the solution to the inhomogeneous wave equation

$$u_{tt} = u_{xx} + f(x,t) \quad u(x,0) = 0 \quad u_t(x,0) = 0$$

using your result from (i).

Problem 4: Second Order Equations - Existence and Classification

(a) For the equation

$$u_{xx} + 2yu_{xy} + y^2u_{yy} = 0$$

classify it by type and determine its canonical form.

(b) State the Cauchy-Kowalevski theorem and indicate (with justifications) what it tells you about the existence of solutions of the initial value problem for the heat equation.

Problem 5. Uniqueness Suppose $\vec{v}(x,t)$ is a smooth vector field, and Ω a domain in R^2. Consider the equation

$$\begin{aligned} u_t &= \Delta u - \vec{v} \cdot \nabla u \\ u(x,0) &= u_0(x) \\ u|_{\partial\Omega} &= f \end{aligned}$$

Prove that classical solutions of the above are unique by proving an appropriate maximum principle.