Comprehensive Exam in PDE's - May 2006

Solve 4 problems. Each problem is 25 points.

1) *First order equations*

Consider the inviscid Burger's equation: $u_t + uu_x = 0$ with initial data

$$u(t=0,x) = \begin{cases} 1 & \text{for } x \leq 0 \\ 1 - x^2 & \text{for } 0 < x \leq 1 \\ x^2 - 4x + 3 & \text{for } 1 < x < 2 \\ -1 & \text{for } 2 \leq x. \end{cases}$$

(i) Is the problem well posed? (State a theorem and verify whether it applies to this Cauchy problem)

(ii) Find the classical solution of the problem (an implicit form is sufficient). For how long is it valid?

(iii) Knowing that Burger's equation comes from the following conservation law $u_t + (0.5u^2)_x = 0$ find the weak solution of the problem past time $t = 1$.

2) *Second order equations*

Consider the following equation in 2 independent variables $u_{xx} - 4u_{xy} + 3u_{yy} = 8x,\ x, y \in \mathbb{R}$

(i) What type of equation is it?

(ii) Compute its canonical form.

(iii) Find its general solution (the one dependent on two arbitrary functions).

3) *Heat equation*

Let $U = \mathbb{R}^n \times [0, T]$. Let $u \in C^2(U)$ be *bounded* and satisfying $u_t - \Delta u \leq 0$ in U. Prove the following weak maximum principle in unbounded domains:

$$M := \sup_{(x,t) \in U} u(x,t) = \sup_{x \in \mathbb{R}^n} u(x,0) =: m.$$

Hint: Fix $x_0 \in \mathbb{R}^n$ and show that $v_{x_0}(x,t) := u(x,t) - \varepsilon(2nt + |x - x_0|^2) \leq m$ for all x, t and $\varepsilon > 0$. Let $\varepsilon \to 0$ to conclude that $u(x_0, t) \leq m$ for all $x_0 \in \mathbb{R}^n$.

4) *Energy method*

a) Let Ω be a smooth bounded domain in $\mathbb{R}^n$. Let u be a C^2 solution of the wave equation

$$\begin{aligned} u_{tt} &= \Delta u, \quad x \in \Omega, t > 0, \\ u(x,t) &= 0, \quad x \in \partial\Omega, t > 0. \end{aligned}$$

Show that the energy

$$E_\Omega(t) := \frac{1}{2}\int_\Omega (u_t^2 + |\nabla u|^2)dx$$

is constant in time.

b) Show the uniqueness of the solutions of the equation:

$$\begin{aligned} &u_{tt} = \Delta u + f(x,t), \quad x \in \Omega, t > 0, \\ &u(x,t) = g(x), \quad x \in \partial\Omega, t > 0, \\ &u(x,0) = h_1(x), \quad u_t(x,0) = h_2(x), \quad x \in \Omega. \end{aligned}$$

5) *Mean value property*

Let $u \in C^2(\mathbb{R}^n)$, $n \geq 2$. Prove that u is harmonic if and only if

$$u(x) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} udS$$

for all $x \in \mathbb{R}^n$, $r > 0$. Here $|\partial B(x,r))|$ denotes the total surface area of the sphere $\partial B(x,r)$. Also, discuss the analogous statement in one dimension.