Comprehensive Exam **PDEs Math 553** **January 2017**

Total points 100. Do 4 out of the 5 problems.

Instructions. Show all your work and make your explanations as full as possible.
Calculators are not allowed on this exam, and neither are books or notes.

Problem 1 (25 points)

Let $G(u) = \frac{1}{2}u^2$. Solve the conservation law $G'(u)u_x + u_t = 0$ for $0 \leq t \leq 3$, with initial condition

$$h(x) = u(x,0) = \begin{cases} 0 & \text{for } x \leq 0, \\ 1 & \text{for } 0 < x \leq 1 \\ -x + 2 & \text{for } 1 < x < 2 \\ 0 & \text{for } 2 \leq x. \end{cases}$$

Problem 2 (25 points)

a) Suppose that $u(x,t)$ solves the wave equation in 3-dimensions, $u_{tt} = c^2 \Delta u$, with initial conditions $u(x,0) = 0$, and $u_t(x,0) = h(x)$, h being a smooth function with compact support. Consider the spherical mean of $u(x)$ on $\mathbb{R}^3$ on a sphere of radius r and center x:

$$M_u(x,r) = \frac{1}{4\pi} \int_{|\xi|=1} u(x + r\xi)\, d\xi$$

where $d\xi$ is the area element of the sphere. Fix $x \in \mathbb{R}^3$. Show that $rM_u(x,r)$ solves the homogeneous 1-dimensional wave equation (with variables r and t).

b) Deduce that

$$u(x,t) = \frac{t}{4\pi} \int_{|\xi|=1} h(x + ct\xi)\, d\xi.$$

c) Show that there exists a constant C so that

$$|u(x,t)| \le \frac{C}{t}$$

for all $x \in \mathbb{R}^3$ and $t > 0$.

Problem 3 (25 points)

For what follows we define

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} f(x)\ e^{-ix\xi}\ dx$$

and

$$f(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} \widehat{f}(\xi)\ e^{ix\xi}\ d\xi$$

for smooth and decaying functions. The Hilbert transform $\mathcal{H}$ is a linear operator that arises in many areas of mathematics including complex and harmonic analysis and the study of water waves. On the Fourier side the Hilbert transform can be defined as

$$\widehat{\mathcal{H}f}(\xi) = -isgn(\xi)\widehat{f}(\xi)$$

$$sgn(\xi) = \begin{cases} -1 & \text{for } \xi < 0 \\ 0 & \text{for } \xi = 0 \\ 1 & \text{for } \xi > 0. \end{cases}$$

Solve the initial value problem

$$\begin{cases} u_t + \mathcal{H}(u_x) = 0, & x \in \mathbb{R}, \quad t \in (0, \infty) \\ u(x,0) = u_0(x) \in \mathcal{S}. \end{cases} \tag{1}$$

Here $\mathcal{S}$ denotes the Schwartz class of smooth and rapidly decreasing functions on $\mathbb{R}$. Express the solution in a closed form as a convolution with a kernel. Show that the above equation obeys a maximum/minimum principle

$$\inf_{x\in\mathbb{R}} u_0(x) \leq u(x,t) \leq \sup_{x\in\mathbb{R}} u_0(x).$$

Problem 4 (25 points)

For $x \in \mathbb{R}^2$, let

$$u(x) = \begin{cases} |x|^2 - 1 & \text{if } |x| \leq 1, \\ \ln(|x|^2) & \text{if } |x| > 1, \end{cases} \qquad f(x) = \begin{cases} 4 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

Show that $\Delta u = f$ weakly in $\mathbb{R}^2$. (Note: here $x = (x_1, x_2)$ and $|x| = \sqrt{x_1^2 + x_2^2}$.)

Problem 5 (25 points)

Let Ω be a bounded domain in $\mathbb{R}^n, n \geq 2$, with smooth boundary. Assume $u(x,t) \in C^\infty(\overline{\Omega} \times [0,T])$ solves the following initial value problem for the heat equation:

$$\begin{aligned} u_t &= \Delta u, && x \in \Omega, \quad t > 0, \\ u(x,0) &= g(x), && x \in \Omega, \\ u(x,t) &= 0, && x \in \partial\Omega, \quad t > 0, \end{aligned}$$

where $g \in C_0^\infty(\Omega)$ (that is g is a smooth function compactly supported in Ω) is given, $g \not\equiv 0$. Assume $g(x) \leq 0$ for all $x \in \Omega$.

(a) Show that $u(x,t) \leq 0$ for all $x \in \Omega, 0 < t < T$.

(b) Actually $u(x,t) < 0$ for all $x \in \Omega, 0 < t < T$. Assuming this, explain why it means that the heat equation allows "infinite propagation speed" of disturbances.