Comprehensive Exam **PDEs Math 553** **January 2014**

Total points 100. Do 4 problems.

Instructions. Show all your working and make your explanations as full as possible. Calculators are not allowed on this exam, and neither are books or notes.

Problem 1 (25 points)

Consider the quasilinear equation $uu_x + u_y = 0$, with initial condition

$$u(x,0) = h(x) = \begin{cases} 0 & \text{for } x < 0 \\ kx - k & \text{for } x > 0, \end{cases}$$

where the number $k > 0$ is given. There is a weak solution $u(x,y)$ that has a jump discontinuity along a curve $x = \xi(y)$. Find this curve and describe the weak solution.

Problem 2 (25 points)

a) (7 points) Consider the fundamental solution of the linear heat equation on $\mathbb{R}^n$,

$$k(x,t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0, \end{cases}$$

Show that for all $t > 0$,

$$\int_{\mathbb{R}^n} k(x,t)\ dx = 1.$$

b) (9 points) Verify that for $n \geq 3$ the fundamental solutions of the heat and the Laplace equation are related by

$$\int_0^\infty k(x,t)\ dt = \frac{1}{\omega_n(n-2)}|x|^{2-n},$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

c) (9 points) Solve explicitly the second order linear parabolic partial differential equation

$$\begin{aligned} u_t &= \Delta u + \vec{a} \cdot \nabla u + bu \\ u(x,0) &= g(x),\ \ (x,t) \in \mathbb{R}^n \times \mathbb{R}_+, \end{aligned} \tag{1}$$

where $b \in \mathbb{R}$ and $\vec{a} \in \mathbb{R}^n$ are constants and g is a bounded and continuous function.

Hint: Recall the definition of the gamma function for $x > 0$,

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\ dt.$$

You may use the fact that $\Gamma(x+1) = x\Gamma(x)$ and that $\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$.

Problem 3 (25 points)

(Liouville's Theorem) a) (5 points) State without proof the formula for the solution of the Dirichlet problem for Laplace's equation on the ball $B(R)$, centered at the origin. That is solve

$$\begin{aligned} \Delta u &= 0, \text{ in } B(R) \\ u &= g, \text{ on } \partial B(R). \end{aligned} \tag{2}$$

b) (10 points) Suppose that u is harmonic and non negative on $\mathbb{R}^n$. Show that for each $R > 0$,

$$\frac{R - |\xi|}{(R + |\xi|)^{n-1}} R^{n-2} u(0) \le u(\xi) \le \frac{R + |\xi|}{(R - |\xi|)^{n-1}} R^{n-2} u(0)$$

for every $|\xi| < R$.

c) (5 points) use part b) to prove that $u(\xi) = u(0)$ for all $\xi \in \mathbb{R}^n$, so that u is constant.

d) (5 points) Prove that if u is harmonic and bounded from below on $\mathbb{R}^n$, then u is constant.

Problem 4 (25 points)

a) (10 points) Consider the diffusion equation

$$\begin{aligned} &u_t = u_{xx}, \text{ in } 0 < x < 1, \quad 0 < t < \infty, \\ &u(0,t) = u(1,t) = 0, \\ &u(x,0) = 4x(1-x). \end{aligned} \tag{3}$$

Show that $0 < u(t,x) < 1$ for all $t > 0$ and $0 < x < 1$.

b) (7 points) Show that $u(x,t) = u(1-x,t)$ for all $t \geq 0$ and $0 \leq x \leq 1$.

c) (8 points) Show that $\int_0^1 u^2(x,t)\, dx$ is a strictly decreasing function of t.

Problem 5 (25 points)

(Kirchhoff) a) (12 points) Suppose that $u(x,t)$ solves the wave equation in 3-dimensions, with initial conditions $u(x,0) = 0$, and $u_t(x,0) = h(x)$. Consider the spherical mean of $u(x)$ on $\mathbb{R}^3$ on a sphere of radius r and center x:

$$M_u(x,r) = \frac{1}{4\pi} \int_{|\xi|=1} u(x + r\xi)\, d\xi$$

where $d\xi$ is the area element of the sphere. Fix $x \in \mathbb{R}^3$. Show that $rM_u(x,r)$ solves the homogeneous 1-dimensional wave equation (with variables r and t).

(13 points) Deduce that

$$u(x,t) = \frac{t}{4\pi} \int_{|\xi|=1} h(x + ct\xi)\, d\xi.$$