Comprehensive Exam — PDEs (Math 553) — January 2008

Total points 100. Do 4 problems.

Instructions Show ALL your working and make your explanations as full as possible. Calculators are not allowed on this exam, and neither are books or notes.

Notation Unless otherwise stated, $T > 0$ is fixed and Ω denotes a smoothly-bounded domain in $\mathbb{R}^n, n \geq 2$. You may use Green's Formulas:

$$\int_\Omega [u\Delta v + \nabla u \cdot \nabla v]\, dx = \int_{\partial\Omega} u \frac{\partial v}{\partial \nu}\, dS$$

$$\int_\Omega [u\Delta v - v\Delta u]\, dx = \int_{\partial\Omega} \left[u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right] dS$$

(1) (25 points) *(First order equations)*

Let $G(z) = -\frac{1}{3}z^3$. Solve the conservation law $G(u)_x + u_y = 0$ for $x \in \mathbb{R}, 0 < y < 3$, given initial data

$$u(x,0) = \begin{cases} 0 & \text{for } 0 < x < 1, \\ 1 & \text{otherwise.} \end{cases}$$

(A well–labeled sketch of the characteristics is a good way to present your solution.)

Then evaluate the jump condition for the shock slope $x'(y)$ when $y > 3$. (You do not need to solve for $x(y)$.)

(2) (25 points) *(Heat equation in unbounded domain)* Consider the initial value problem for the heat equation

$$u_t - u_{xx} = 0, \quad \text{in } \mathbb{R} \times (0, \infty),$$
$$u(x,0) = u_0(x), \quad \text{on } \mathbb{R} \times \{t = 0\},$$

where u_0 is smooth with compact support.

(a) Use the Fourier transform to deduce the solution

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4t}} u_0(y)\, dy,$$

valid for all $t > 0$.

Hint. To invert the Fourier transform, use the identity $\int_{\mathbb{R}} e^{ia\xi - b\xi^2}\, d\xi = (\frac{\pi}{b})^{\frac{1}{2}} e^{-\frac{a^2}{4b}}$.

(b) Show that $u(x,t) - u_0(x) \to 0$ in $L^2(\mathbb{R})$ as $t \downarrow 0$. *Hint.* The Fourier transform preserves the L^2 norm.

(3) (25 points) *(Heat equation in bounded domain)*

Assume $u(x,t)$ is smooth on $\overline{\Omega \times [0,T]}$ and solves:

$$\begin{aligned} u_t &= \Delta u, && x \in \Omega, \quad t > 0, \\ u(x,0) &= g(x), && x \in \Omega, \\ u(x,t) &= 0, && x \in \partial\Omega, \quad t > 0, \end{aligned}$$

where $g \in C^\infty(\overline{\Omega})$ satisfies $g(x) \leq 0$ for all x.

(a) Use a major theorem to explain why $u(x,t) \leq 0$ for all $x \in \Omega, 0 < t < T$.

(b) Suppose in addition g has compact support in Ω, and $g(x) < 0$ for some $x \in \Omega$. It can be shown $u(x,t) < 0$ for all $x \in \Omega, 0 < t < T$.

Use this fact to justify the claim that "the heat equation has infinite propagation speed".

(4) (25 points) *(Wave equation)*

Let $\gamma > 0$ be constant, and assume $u(x,t)$ is smooth on $\overline{\Omega \times [0,T]}$. Consider the wave equation with friction,

$$u_{tt} = c^2 \Delta u - \gamma u_t,$$

with boundary condition $u(x,t) = 0$ when $x \in \partial\Omega, t \geq 0$.

(a) Show the energy $E(t) = \frac{1}{2}\int_\Omega (u_t^2 + c^2|\nabla u|^2)\,dx$ is dissipated, that is, $E'(t) \leq 0$.

Remark. If you cannot do this in n dimensions, then do it in one dimension on the interval $\Omega = [a,b]$ (for half-credit).

(b) State a uniqueness result for the initial-and-boundary-value problem for this wave equation with friction. Prove it using part (a).

(5) (25 points) *(Poisson's equation)*

(a) Define what it means to say $\Delta u = f$ weakly in $\mathbb{R}^3$.

(b) For $(x,y,z) \in \mathbb{R}^3$, write $r = \sqrt{x^2+y^2+z^2}$ and define

$$u(x,y,z) = \begin{cases} 3 - r^2 & \text{if } r \leq 1, \\ 2/r & \text{if } r \geq 1, \end{cases} \qquad f(x,y,z) = \begin{cases} -6 & \text{if } r \leq 1, \\ 0 & \text{if } r > 1. \end{cases}$$

Show $\Delta u = f$ classically when $r \neq 1$.
Then show $\Delta u = f$ weakly in $\mathbb{R}^3$.