Comprehensive Exam in PDE's - January 2007

The exam consists of 5 problems. Each problem is worth 25 points. You must do four out of five problems. You will receive credit for the best four. You must show all work to receive full credit.

Unless otherwise specified you may assume that all functions are well-behaved (C^∞ and rapidly decaying.) In general you may use any analytical facts you need as long as you state them clearly (and correctly). Some potentially useful identities are listed below.

Table of (possibly useful) Identities

$$\hat{\phi}(k) = \mathcal{F}(\phi) = \int e^{-2\pi ikx}\phi(x)dx$$

$$\phi(x) = \mathcal{F}^{-1}(\hat{\phi}) = \int e^{2\pi ikx}\hat{\phi}(k)dk$$

$$\mathcal{F}(\phi * \psi) = \hat{\phi}(k)\hat{\psi}(k)$$

$$\int e^{2\pi ikx}e^{-a|k|}dk = \frac{2a}{a^2+4\pi^2x^2}$$

$$\int e^{-2\pi ikx}\frac{2a}{a^2+4\pi^2k^2}dk = e^{-a|x|}$$

$$\int e^{2\pi ikx}e^{-a^2x^2}dx = \frac{\sqrt{\pi}}{a}e^{-\frac{\pi^2k^2}{a^2}}$$

$$\int e^{2\pi ikx}\frac{\chi_{[-\pi,pi]}(k)}{\sqrt{1-4\pi^2k^2}}dk = J_0(x)$$

Where χ is the usual characteristic function and J_0 the standard Bessel function.

Problem 1: Consider the Helmholtz equation in $\mathbf{R}^3$:

$$-\Delta u + m^2 u = 0.$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. The goal of this problem is to find a fundamental solution K for the Helmholtz equation solving

$$-\Delta K + m^2 K = \delta(x).$$

Here $\delta(x)$ is the delta distribution and K solves the above equation in the sense of distributions.

(1) State precisely what it means for K to solve the above equation in the sense of distributions.

(2) As in the case of the Laplace equation the invariance of the equation under rotations suggests looking for a fundamental solution that is itself rotationally invariant (radial). Write the Helmholtz equation $-\Delta K + \alpha^2 K = 0$ in spherical coordinates and solve it.

HINT: Make the change of variable $K = H/r$. Choose the solution which decays as $r \to \infty$.

(3) Show that your solution satisfies the distributional formulation in (1).

Problem 2: In this problem $\mathcal{F}$ will denote the Fourier transform and $\mathcal{F}^\dagger$ the inverse transform, as denoted on the cover of the exam. If you use a different convention please denote this clearly on your exam.[1]

The Hilbert transform $\mathbf{H}$ is a linear operator that arises in many areas of mathematics, including complex and harmonic analysis and the study of water waves. One way to define the Hilbert transform is via the Fourier transform: if $\mathcal{F}$ represents the Fourier transform and $\mathcal{F}^\dagger$ the inverse Fourier transform then the Hilbert transform can be defined as

$$\mathbf{H}(\phi) = \mathcal{F}^\dagger(\text{sgn}(k)\mathcal{F}(\phi))$$

where $\text{sgn}(k)$ is the sign of k:

$$\text{sgn}(k) = \begin{cases} -1 & k < 0 \\ 0 & k = 0 \\ 1 & k > 0 \end{cases}$$

Consider the evolution

$$u_t = i\mathbf{H}(u_x) \qquad u(x,0) = u_0(x)$$

(1) Solve the above initial value problem via Fourier transform. Write your solution in the form

$$u(x,t) = \int_{-\infty}^{\infty} K(x-y,t)u_0(y)dy$$

compute the kernel $K(\cdot,t)$ as completely as you are able[2].

(2) Show that if the initial data u_0 is bounded then the above equation has a maximum/minimum principle

$$\inf_{x\in\mathbf{R}} u_0(x) \le u(x,t) \le \sup_{x\in\mathbf{R}} u_0(x)$$

[1]and do so at your own risk.....

[2]It can be expressed in a reasonably simple closed form.

Problem 3: (a) Define the weak solutions of

$$u_t + u^2 u_x = 0$$

Find a weak solution with initial data:

$$u(0,x) = \begin{cases} 1 & x \leq 0 \\ 2 & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

for small times $t > 0$.

(b) Consider the conservation law:

$$\left(\frac{u^2}{2}\right)_t + \left(\frac{u^4}{4}\right)_x = 0$$

Show that smooth solutions $u > 0$ of this equation coincide with the ones in part (a). Find a weak solution (for small time $t > 0$) of this equation with the same initial condition as in (a).

Problem 4: Define the weak solutions of the equation: $u_{tt} - u_{xx} = 0$. Find the weak solution of the above equation which satisfies the initial condition:

$$u(0,x) = u_0(x) = \begin{cases} 0 & x \le -1 \\ 1 & -1 < x < 1 \\ 0 & x \ge 1 \end{cases} \qquad u_t(0,x) = 0.$$

Problem 5

Consider the heat equation on an interval I for $t \geq 0$ with Dirichlet boundary conditions

$$u_t = u_{xx} - tu \quad u(x,0) = u_0(x) \qquad u|_{\partial I} = 0$$

assume that u is in $C^2(I \times (0,T)) \cap C(\overline{I \times (0,T)})$

(1) Show that u has no interior local minima where u is negative and no interior local maxima where u is positive.

(2) Conclude that u has a unique solution.