Comprehensive Exam in PDE's - January 2006

Each probelm is worth 25 points. It suffices to solve 4 problems to get full credit.

1) *First order equations*

Consider the following initial value problem:

$$u_t + (\frac{1}{2}u^2)_x = 0$$
$$u(0,x) = u_0(x)$$

where

$$u_0(x) = \begin{cases} 0 & \text{for } x \le 0 \\ 1 & \text{for } 0 < x \le 1 \\ -x+2 & \text{for } 1 < x < 2 \\ 0 & \text{for } 2 \le x. \end{cases}$$

Find the solution valid until time $t = 3$.

2) *Distributions*

i) Let $u(x) = |x|$ on $\mathbb{R}$. Calculate the first and second weak derivatives.

ii) Solve the equation

$$u_t - u_{xx} + u_x = 0, \quad t > 0, x \in \mathbb{R}$$
$$u(0,x) = f(x) \in \mathcal{S}'$$

using Fourier transform. You may find the following integral useful:

$$\int_{-\infty}^{\infty} e^{-y^2} e^{isy} dy = \sqrt{\pi} e^{-s^2/4}.$$

3) *Energy Method*

Consider the equation

$$u_{tt} + u_{xxxx} = 0, \quad t \ge 0, 0 \le x \le 1$$
$$u(t,0) = u(t,1) = 0$$
$$u_x(t,0) = u_x(t,1) = 0.$$

i) Find an appropriate conserved quantity. Prove your claim.

ii) Using part i) prove that the given equation with the initial conditions $u(0,x) = f(x)$, $u_t(0,x) = g(x)$ has a unique solution.

4) *Maximum principle*

State and prove a maximum principle for the equation ($\gamma \in \mathbb{R}$)

$$u_t - u_{xx} + \gamma u_x = 0, \quad t \geq 0, a \leq x \leq b.$$

5) *Laplace equation*

i) Find the Green function for the Dirichlet problem on the ball of radius 1 in $\mathbb{R}^3$. (The fundamental solution for the Laplacian on $\mathbb{R}^3$ is $K(x) = -(4\pi|x|)^{-1}$.)

ii) Write down and justify the formula (Poisson) for smooth solutions of:

$$\begin{aligned} \Delta u &= 0 \qquad x \in \mathbb{R}^3,\ |x| < 1 \\ u(x) &= g(x) \qquad \text{for } |x| = 1 \end{aligned}$$

where g is a smooth function on the unit sphere.