Comprehensive Exam in PDE's - August 2007

Problem 1:

Let $G(\rho) = \rho(1 - \rho/\rho_{\max})$, where $\rho_{\max}$ is a given constant. Using the method of characteristics for quasilinear first order equations, solve

$$G(\rho)_x + \rho_y = 0 \qquad \text{for } x \in \mathbf{R}, 0 < y < 2,$$

given initial data

$$\rho(x,0) = \begin{cases} \rho_{\max}/2 & \text{for } -1 < x < 0, \\ 0 & \text{otherwise.} \end{cases}$$

A good way to present the solution is to sketch the projected characteristics. Be sure to show any shock curves or rarefaction fans, with justification. You are *not* required to find a *formula* for the whole solution, though.

(ii) What happens at $y = 2$ that changes the solution? (Give a one sentence answer.)

Problem 2 (25 points)

Suppose u is harmonic and non-negative on $\mathbf{R}^n$.

i) Show, by using the Poisson formula for the ball, that for each $R > 0$,

$$\frac{R^{n-2}(R-|\zeta|)}{(R+|\zeta|)^{n-1}}u(0) \leq u(\zeta) \leq \frac{R^{n-2}(R+|\zeta|)}{(R-|\zeta|)^{n-1}}u(0) \quad \text{for } |\zeta| < R.$$

ii) Show that if u is harmonic and non-negative on $\mathbf{R}^n$ then u must be constant.

iii) Use (ii) to deduce the following slight generalization of Liouville's theorem: If $v : \mathbf{R}^n \to \mathbf{R}$ is harmonic and either bounded from below or from above, then v is constant.

Hint: The Poisson kernel is given by

$$K(x,\xi) = \frac{R^2 - |\xi|^2}{R\omega_n |x-\xi|^n}$$

where ω_n is the area of the (n-1) sphere.

Problem 3. An Inverse Wave Problem

Let $G(x,t) = \frac{1}{2}\chi_{[-t,t]}(x)H(t)$. Recall that $\chi_{\mathbf{A}}$, the characteristic function of a set A is defined to be

$$\chi_{\mathbf{A}}(x) = \begin{array}{ll} 1 & x \in \mathrm{A} \\ 0 & x \notin \mathrm{A} \end{array}$$

and the Heaviside function $H(t)$ to be

$$\begin{aligned} H(t) &= 1 \quad t > 0 \\ &= 0 \quad t \leq 0 \end{aligned}$$

(i) Show that (in the sense of distributions) $G(x,t)$ solves

$$G_{tt} - G_{xx} = \delta(x)\delta(t)$$

Be sure to state clearly what it means for G to satisfy the above equation in the sense of distributions. You will find it easier to work in characteristic coordinates.

(ii) Show that

$$U(x,t) = \int G(x - x', t)g(x')dx'$$

is a classical solution of

$$U_{tt} - U_{xx} = 0$$

if g is C^1. What initial conditions does it satisfy?

(iii) Using (i) and (ii) derive the D'Alembert solution to the one dimensional wave equation on the whole line for general initial data.

Problem 4: Maximum Principle

(i) Suppose that $u(x,t)$ satisfies a linear PDE with vanishing boundary conditions, and that this evolution is NOT positivity preserving - there exists initial data $f(x)$ - $u(x,0) \geq 0$ such that $u(x,t) < 0$. Show that the linear PDE does not satisfy the maximum principle.

(ii) Solve the equation

$$u_t = -\left(\frac{\partial^{2008} u}{\partial x^{2008}} + \frac{\partial^{2008} u}{\partial y^{2008}}\right) \qquad u(x,y,0) = g(x,y)$$

via Fourier tranform. Write your solution in the form of a convolution of the initial data $g(x,y)$ with a kernel $K(x,y,t)$.

(iii) Show that a necessary condition for the above equation to be positivity preserving is that the kernel $K(x,y,t)$ be strictly positive.

(iv) Show that the kernel $K(x,y,t)$ is not a strictly positive function and conclude that the evolution above does not have a maximum principle.

Hint: Consider $\int x^k K(x,y,t)dxdy$ for appropriate k? What does this tell you about the kernel?