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Comprehensive Exam in PDE's - August 2006

Each problem is worth 25 points. It suffices to solve 4 problems to get full credit.

1) *General nonlinear first order equations*

Consider $u = u_x^2 + u_y^2$ with the initial condition $u(x,0) = ax^2$. For which positive constants "a" is there a solution? Is it unique? Find all solutions.

2) *Hyperbolic equations*

Consider the equation $u_{tt} - u_{xx} + xu_x + u = 0$.

i) Write down the definition of a weak solution of the equation.

ii) Consider the initial data

$$u(t=0,x) = \begin{cases} 1, & x \le 0 \\ 0, & x > 0 \end{cases}$$

$$u_t(t=0,x) = 0$$

where do you expect the weak solution to be discontinuous? Carefully explain your answer.

iii) Find the transmission condition across one characteristic.

3) *Maximum principle*

State and prove a maximum principle for the equation ($\gamma \in \mathbb{R}$)

$$u_t - u_{xx} + \gamma u_x = 0, \quad t \ge 0, a \le x \le b.$$

4) *Energy Method*

Consider the equation

$$\begin{aligned} u_t - \Delta u = f, &\quad x \in \Omega,\ 0 < t < T, \\ u = 0, &\quad x \in \partial\Omega,\ 0 < t < T, \\ u = g, &\quad x \in \Omega,\ t = 0. \end{aligned}$$

Using energy methods, state and prove a result about "continuous dependence on the data" for the given equation. You may assume that all functions are smooth on $\bar{\Omega}$ and that g has compact support in Ω.

5) *Poisson's equation*

Let f be a smooth function with compact support in $\mathbb{R}^3$.

i) Write down the fundamental solution of the Laplacian in $\mathbb{R}^3$ (solving $\Delta K = \delta$ in $\mathbb{R}^3$).

ii) Find u solving $\delta u = f$ in $\mathbb{R}^3$ and satisfying $u(x) \to 0$ as $|x| \to \infty$.

iii) Assume that f is radially symmetric and is supported in the unit ball. Suppose also that $\int_{\mathbb{R}^3} f(x)dx = 1$. Show that $u(x) = K(x)$ for all $|x| > 1$.