550 Comprehensive Exam — May 2009

1. Let $p(z)$ be a complex quadratic polynomial, i.e.

$$p(z) = az^2 + bz + c,$$

where $a, b, c \in \mathbb{C}$, and consider the equation

$$\frac{dz}{dt} = p(z(t)), \quad z(0) = z_0 \in \mathbb{C}. \tag{1}$$

We are interested in the set of polynomials such that every solution of (1), for any $z_0 \in \mathbb{C}$, lies in $\mathbb{C}$ for all time. Define this set of polynomials by $\mathcal{P}$. Write down necessary and sufficient conditions on a, b, c so that $p \in \mathcal{P}$. If $p \in \mathcal{P}$, what kind of zeros can p have?

2. Consider the initial value problem (IVP)

$$(1 - 2t)\, x'(t) + \alpha_1 x(t) = \alpha_2, \tag{2}$$
$$x(0) = A, \tag{3}$$

where α_1, α_2, A are real constants and $\alpha_1 > 0$. We are aiming to show that this IVP has infinitely many solutions for $t \in [0, 1]$.

   (a) Show that the IVP (equations (2) and (3) together) has a unique solution for $t \in [0, 1/2)$. Determine this solution.

   (b) Show that (2), plus the initial condition $x(1) = B$, has a unique solution for $t \in (1/2, 1]$. Determine this solution.

   (c) Show that the boundary value problem

$$(1 - 2t)\, x'(t) + \alpha_1 x(t) = \alpha_2, \quad x(0) = A, \quad x(1) = B$$

   has a *continuous* solution for any real B, and compute this solution. (From this it follows that the original IVP has infinitely many solutions, one for each B.)

3. Consider the system $\dfrac{dx}{dt} = Ax$, where

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{pmatrix}.$$

Show that the solutions to this system all approach the origin as $t \to \infty$.

Hint. Write $A = B - 2I$, where

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

What are the eigenvectors of B?

4. The equation for the nonlinear pendulum is given by

$$\ddot{\theta} + \sin(\theta) = 0,$$

where θ is the angle between the pendulum and the vertical.

   (a) Show that this system has a conserved quantity and determine it.

(b) Write this equation as a system of first-order ODEs and determine the location and type of all the fixed points of this system.

(c) Draw the level sets of the conserved quantity; use symmetry to argue that this system has a family of periodic orbits around one of the fixed points.

(d) Finally, if we add friction to this system, i.e. modify the equation to

$$\ddot{\theta} + c\dot{\theta} + \sin(\theta) = 0$$

for some $c > 0$, how does this change the system (to wit, how do the types of the fixed points change when the friction is added)?