MATH 550 COMPREHENSIVE EXAMINATION
May 2007

ORDINARY DIFFERENTIAL EQUATIONS AND DISCRETE MAPS
DO ALL PROBLEMS.

1. EXISTENCE AND UNIQUENESS OF SOLUTIONS TO INITIAL VALUE PROBLEMS
   Let $f(x) = x^{2/3}\sin(1/x)$ for all $x \neq 0$ and $f(0) = 0$.
   a. Show that the initial value problem $dx/dt = f(x)$, $x(0) = 0$ has a unique solution.
   b. Are solutions of the initial value problem with $x(0) \neq 0$ defined for all time?
   c. Show that all solutions of the initial value problem with $x(0) \neq 0$ are bounded.

2. LINEAR DIFFERENTIAL EQUATIONS
   Let $f(X) = AX$ where A is a 2 x 2 matrix of real constants and X is a vector.
   a. Write down the three **real** canonical forms for a 2 x 2 real matrix A.
   b. Write down the solution forms for the initial value problem $dX/dt = f(X)$, $X(0) = X_0$ using the three real canonical forms in part a.
   c. Does this differential equation above generate a flow on the plane?

3. AREA PRESERVING MAP
   Consider the following map f of the 2-torus T^2 defined by
   $$f(x, y) = (y, x + y) \bmod 1 \text{ for } 0 \leq x, y < 1.$$
   a) Show that f is an area-preserving, orientation reversing map.
   b) Show that there is a countable infinity of periodic points of f.
   c) Show that the periodic points of f are dense in T^2.

4. GRADIENT FLOWS AND AREA-PRESERVING FLOWS
   Consider the following system of first order ordinary differential equations:
   $$dx/dt = f(x, y) \text{ and } dy/dt = g(x, y)$$
   where f and g are of class C^∞ and satisfy $\partial f/\partial x + \partial g/\partial y = 0$ for all (x,y) in the plane.
   a. Show that there is a C^∞ real valued function F such that
   $$f = \partial F/\partial y \text{ and } g = -\partial F/\partial x.$$
   b. Show that the local flow of the vector field orthogonal to (f, g) is irrotational; that is, the orthogonal vector field is the gradient field of a real valued function G.
   c. If f and g are defined by $f(x, y) = 3x^2y^2$ and $g(x, y) = -2xy^3 - 3$, find G(x, y).