MATH 550 Comprehensive Exam ODEs

Do **FOUR** of the following problems.

Problem # 1 Consider the equation $\dot{x} = Ax$, where $x \in \mathbb{R}^2$. Let λ_1, λ_2 be eigenvalues of A. Assume that the eigenvalues have zero real parts $\Re(\lambda_1) = \Re(\lambda_2) = 0$.

- Is the equilibrium $x = 0$ stable for any such A? If yes, then prove it. If no, then give a counterexample.
- If the answer to the above question is no, then what additional conditions must be imposed on A so the equilibrium point $x = 0$ is Lyapunov stable. Justify your answer.

Problem # 2 Let $\dot{x} = f(x)$ be an ODE with a rest point $x = 0$. Let V be a C^∞ smooth Lyapunov function, *i.e.* $V(0) = 0, V(x) > 0$ if $x \neq 0$, and $\nabla V \cdot f(x) \leq 0$. Prove that the equilibrium $x = 0$ is Lyapunov stable.

Problem # 3 Consider the equation $\dot{x} = x^\alpha, t \geq 0, x \geq 0$.
For which $\alpha \in [0, \infty)$, does the equation have a unique solution with $x(0) = 0$?
For which $\alpha \in [0, \infty)$ can any solution be extended to all time?
Explain your answers. Either refer to appropriate theorems or prove your conclusions.

Problem # 4 Consider the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + x^3. \end{aligned}$$

- Find all the critical points and their stability type.
- Sketch the phase portrait of the system.

Problem # 5 Consider the linear system with periodic coefficients

$$\dot{x} = A(t)x,$$

where $x \in \mathbb{C}^n$. Prove that the fundamental solution $\Phi(t)$ can be represented in the form

$$\Phi(t) = P(t)e^{Ct},$$

where $P(t)$ is periodic matrix and C is a constant matrix.