Math 550 Comprehensive Exam

January 2010

DO ALL FOUR PROBLEMS

1. (a) Given $f \in C^1(\mathbb{R}^2)$, show that the system

$$\dot{X} = f(X)$$

is a Hamiltonian system if and only if

$$\nabla \cdot f(X) = 0$$

for all $X \in \mathbb{R}^2$.

(b) Show that the system

$$\begin{aligned} \dot{x} &= a_{11}x + a_{12}y + Ax^2 - 2Bxy + Cy^2 \\ \dot{y} &= a_{21}x - a_{11}y + Dx^2 - 2Axy + By^2. \end{aligned}$$

is a Hamiltonian system by finding the Hamiltonian function.

(c) Sketch the phase portrait of the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + x^2. \end{aligned}$$

2. Consider the system $\dot{X} = AX$, where

$$A = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 2 & 0 & 6 \end{bmatrix}.$$

(a) Find the stable, unstable and center subspaces E^s, E^u and E^c for this system.

(b) For $X_0 \in E^c$, show that the sequence of points $X_n = e^{An}X_0 \in E^c$.

(c) Solve the system.

3. Show that the system

$$\begin{aligned} \dot{x} &= -x + 2y, \\ \dot{y} &= -x - 2y^3 \end{aligned}$$

has no periodic orbits. Hint: Show that the function $V(x, y) = ax^2 + by^2$ is a strict Lyapunov function for this system for some values of a, b. Determine at least one such pair (a, b). Deduce from this that there are no periodic orbits.

4. Consider the system

$$x'(t) = f(t)x(t),$$

where f is a continuous function with period 1. Compute the *time-1* map of this system, i.e. the function $F_1(x)$ which has the property that $F_1(\xi) = x(1)$, where $x(t)$ solves

$$x'(t) = f(t)x(t), \quad x(0) = \xi.$$

From this, compute a formula for $x(n)$ in terms of f and $x(0)$ for any integer n. What is a sufficient condition on f so that

$$\lim_{t \to \infty} x(t) = 0?$$