MATH 550 COMPREHENSIVE EXAMINATION
January 2008

ORDINARY DIFFERENTIAL EQUATIONS AND DISCRETE MAPS
DO ALL FOUR PROBLEMS

1. AREA PRESERVING MAP
   Consider the following map f of the 2-torus T^2 defined by
   $$f(x, y) = (x + 2y, 2x + 3y) \bmod 1 \text{ for } 0 \le x, y < 1.$$

   a. Show that f is area-preserving.
   b. Is f orientation-preserving?
   c. Show that f has a countable infinity of periodic points.
   d. Show that the periodic points of f are dense in T^2.

2. HAMILTONIAN SYSTEMS AND AREA-PRESERVING FLOWS
   Consider the following system of first order ordinary differential equations:
   $$dx/dt = f(x, y) \text{ and } dy/dt = g(x, y)$$
   where f and g are of class C^∞ and satisfy $\partial f/\partial x + \partial g/\partial y = 0$ for all (x,y) in the plane.

   a. Show that the local flow of the vector field (f, g) is area-preserving.
   b. Show that there is a C^∞ real valued function F (the Hamiltonian) such that
   $$f = \partial F/\partial y \text{ and } g = -\partial F/\partial x.$$
   c. If f and g are defined by $f(x, y) = 3x^2y^2$ and $g(x, y) = -2xy^3 - 3$, find F(x, y).

3. UNIQUENESS/NONUNIQUENESS OF SOLUTIONS OF INITIAL VALUE PROBLEMS

   Let $f(x) = x^{2/3} \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$.

   a. Decide whether the initial value problem $dx/dt = f(x)$, $x(0) = 0$ has a unique solution. If the solution is unique, then prove it. If not, prove it.
   b. Let $x(0) \neq 0$. Can a solution of the initial value problem become unbounded in x as t increases?
   c. Are solutions of the initial value problem with $x(0) \neq 0$ defined for all time?

4. PLANAR FLOWS
   Consider the system of differential equations
   $$dx/dt = x^2 - y^2,\ dy/dt = 2xy,\ x(0) = x_0,\ y(0) = y_0.$$

a. Solve this initial value problem for $x(t, x_0, y_0)$ and $y(t, x_0, y_0)$.
b. Describe the phase curves of the system.
c. Does the system define a flow on the plane? If not on the plane, where is a flow defined?