MATH 550 Comprehensive Exam   Dynamical Systems/ODE

Do **FOUR** of the following five problems.

1. Area preserving maps
   Consider the following map f of the 2-torus T^2 defined by
   $$f(x, y) = (x + y, x + 2y) \bmod 1, \text{ for } 0 \leq x, y < 1.$$

   a) Show that f is area-preserving.
   b) Show that f has a countably many periodic points.
   c) Show that the periodic points of f are dense in T^2.

2. Hamiltonian systems and area-preserving flows
   Consider the following system of first order ordinary differential equations:
   $$dx/dt = f(x, y) \text{ and } dy/dt = g(x, y)$$
   where f and g are of class C^∞ and satisfy $\partial f/\partial x + \partial g/\partial y = 0$ for all (x,y) in the plane.

   a) Show that the vector field (f, g) gives rise to an area-preserving local flow.
   b) Show that there is a C^∞ real valued function F (the Hamiltonian) such that
   $$f = \partial F/\partial y \text{ and } g = -\,\partial F/\partial x.$$

3. Vector fields and flows generated by a differential equation
   Consider the vector field on the plane defined by $(x^2 - y^2, 2xy)$.
   a) Write down the system of differential equations that this vector field defines. Find a first integral of the system.
   b) Does the differential equation system generate a local flow?
   c) Does the differential equation system generate a flow?
   d) Draw the phase portrait for the system.

4. Existence and Uniqueness of Solutions
   a) State an existence and uniqueness theorem for a system of ordinary differential equations.
   b) Prove the theorem stated in a).

5. Existence and Uniqueness of Solutions
   Consider the differential equation on the line given by $dx/dt = f(x)$ where f is defined by $f(0) = 0$ and $f(x) = x \sin(1/x)$ for $x \neq 0$.
   a) Decide whether the initial value problem $dx/dt = f(x)$ and $x(0) = 0$ has a solution.
   b) If the solution is unique, then prove it. If not, state a reason.
   c) If there are many solutions, then exhibit at least two solutions. Are there only two?