MATH 550 COMPREHENSIVE EXAMINATION
August 2006

ORDINARY DIFFERENTIAL EQUATIONS AND DISCRETE MAPS
DO ALL PROBLEMS.

1. Area preserving map
   Consider the following map f of the 2-torus T^2 defined by
   $$f(x, y) = (x + y, x + 2y) \bmod 1 \text{ for } 0 \leq x, y < 1.$$

   a) Show that f is area-preserving.
   b) Show that f has a countable infinity of periodic points.
   c) Show that the periodic points of f are dense in T^2.

2. Hamiltonian systems and area-preserving flows
   Consider the following system of first order ordinary differential equations:
   $$dx/dt = f(x, y) \text{ and } dy/dt = g(x, y)$$
   where f and g are of class C^∞ and satisfy $\partial f/\partial x + \partial g/\partial y = 0$ for all (x,y) in the plane.

   a. Show that the local flow of the vector field (f, g) is area-preserving.
   b. Show that there is a C^∞ real valued function F (the Hamiltonian) such that
   $$f = \partial F/\partial y \text{ and } g = -\partial F/\partial x.$$
   c. If f and g are defined by $f(x, y) = 3x^2y^2$ and $g(x, y) = -2xy^3 - 3$, find F(x, y).

3. Let $f(x) = x^{2/3}$ for for $x \neq 0$ and $f(0) = 0$.
   a. Decide whether the initial value problem $dx/dt = f(x)$, $x(0) = 0$ has a unique solution.
   b. Can any solution of the initial value problem with $x(0) \neq 0$ become unbounded in x as t increases?
   c. Are solutions of the initial value problem with $x(0) \neq 0$ defined for all time?

4. Let $f(x) = x^{2/3} \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$.
   a. Decide whether the initial value problem $dx/dt = f(x)$, $x(0) = 0$ has a unique solution. If the solution is unique, then prove it. If not, prove it.
   b. What conditions does f satisfy in a neighborhood of the origin?
   c. Can any solution of the initial value problem with $x(0) \neq 0$ become unbounded in x as t increases?
   d. Are solutions of the initial value problem with $x(0) \neq 0$ defined for all time?