Students who are taking this exam as the Math 542 Graduate Comprehensive Exam should do all six problems.
Students taking this exam to satisfy the undergraduate complex analysis requirement should only do Problems 1, 2, 3 and 4. Each problem is worth 10 points.
Justify all your answers. Good Luck!

<u>Notation:</u> *D denotes the unit disk in $\mathbb{C}$ and $H(G)$ denotes the set of all analytic functions on a non-empty set $G \subseteq \mathbb{C}$.*

1. Let $U \subseteq \mathbb{C}$ be open, $z_0 \in U$ and $f \in H(U)$ be such that $f'(z_0) \neq 0$. Prove that there is $r > 0$ such that $\frac{2\pi i}{f'(z_0)} = \int_{\gamma_r} \frac{dz}{f(z)-f(z_0)}$, where $\gamma_R(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi$.

2. Let $G \subset \mathbb{C}$ be an open simply connected set and $G \neq \mathbb{C}$. Let $f : G \to G$ be analytic in G and let f be not the identity mapping of G. Prove that f can have at most one fixed point (i.e., a point $z \in G$ such that $f(z) = z$).

3. Let $G = \{z \in \mathbb{C} : |z-2| < 1\}$ and let f be analytic function in the closed disk $\overline{G}$ except for one simple pole z_0 inside. Suppose that $|f(z)| \equiv 1$ on the boundary of G. Prove that for every $a \in \mathbb{C}$, $|a| > 1$, the inverse image $f^{-1}(a)$ contains exactly one point.

4. Use residues to evaluate the following improper integral: $\int_{-\infty}^{+\infty} \frac{\cos x dx}{(x^2+a^2)(x^2+b^2)}$ where $a > b > 0$. **Justify your answer completely** (calculation alone is not sufficient).

5. Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions in $H(D)$ such that $|f(z)| \leq 2018$ for every $z \in D$. Suppose that $\lim_{n\to\infty} f_n\left(\frac{1}{j}\right)$ exists for every $j \in \mathbb{N}$. Prove that $\lim_{n\to\infty} f_n(z)$ exists for every $z \in D$.

6. Let $K = \left\{z \in \mathbb{C} : |z| \leq 1 \text{ and } \left|z - \frac{1}{3}\right| \geq \frac{2}{3}\right\}$ and $G = \operatorname{int}(K)$ (interior of K).
(a) Is it true that polynomials are dense in $H(G)$ (w.r.t. standard uniform convergence on compact subsets of G)? **Justify your answer completely.**
(b) Suppose that f is analytic in a neighborhood of K. Is it true that f can be approximated (uniformly in K) by a sequence of polynomials? **Justify your answer completely.**
(c) Suppose that f is analytic in a neighborhood of K. Is it true that f can be approximated (uniformly in K) by a sequence of the following rational functions $R_n(z) = \sum_{j=0}^{m_n} a_j z^j + \sum_{k=1}^{l_n} \frac{b_k}{z^k}, n \in \mathbb{N}$? **Justify your answer completely.**